


Exact recursive updating of uncertainty sets

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Abstract

This paper addresses the classical problem of determining the set of possible states of a linear discrete-time system subject to bounded disturbances from measurements corrupted by bounded noise. These so-called uncertainty sets evolve with time as new measurements become available. We present an exact, computationally effective algorithm that updates the uncertainty set at some time instant to the uncertainty set at the next time instant.

1 INTRODUCTION

If a linear, time-invariant dynamic system is driven by set-bounded process noise, and has measurements corrupted with set-bounded observation noise, then the set of current possible states of the system consistent with the measurements up to the current time is termed the *state uncertainty set* (or simply *uncertainty set*). This set membership estimation problem is fundamental and has many applications, for example in control under constraints in the presence of noise [3], [9] and model (in)validation, see [24] and [26]. A closely related topic is identification of bounded-parameter models [2], [6] and [22].

The first results on recursive determination of the uncertainty set are in [28] and [32]. Since the appearance of these papers there has appeared an extensive literature on the topic. See [7] and [21] for background on the set-bounded approach to uncertainty, the survey paper [19] and the book [4]. Some of the many other papers which consider this problem are [5], [26], [30] and [31].

Under reasonable assumptions to be made explicit shortly, the uncertainty set at any time is a convex polytope, and hence is determined either by its vertex set, or by a system of inequality constraints. In the algorithm presented in this paper both representations are used; the recursive computational procedure updates the vertices, facets, and vertex-facet incidence matrix of the uncertainty set. The updating process is based on Theorems 9 and 10, and they in turn rely on Theorem 12. Although a precise statement requires some preliminary notation, the basic idea is very simple: any point in the uncertainty set at the current time, along with a direction in its normal cone, is mapped to a point in the uncertainty set at the next time instant, along with a direction in its normal cone. See Fig. 11. Using this idea, which first appeared in [12], the vertex-facet incidence matrix is updated. It is remarkable that, apart from preliminary sorting of the vertices and facets of S_{k-1} , this updating uses only logical operations on $(0, 1)$ matrices. Knowing the combinatorial structure of the updated uncertainty set, that is its incidence matrix, it is then easy to calculate its vertices and facets. For this step linear interpolation and affine updating of vectors are all that is required.

Let us put our idea into historical context. In the first part of the seminal paper [32] an exact in principle primal solution to the problem of recursively determining polytopic uncertainty sets is given. But it requires (Minkowski) addition, and intersections, of polytopes, both of which are time-consuming. Exact, recursive primal methods often use Fourier-Motzkin elimination, see [14], [25] and [29]. In these primal implementations it is the identification of redundant inequality constraints that is most demanding computationally.

In Section V of [32] a dual to the primal approach is presented. Using the theory of conjugate functions an equation describing dynamic evolution of the support function of the uncertainty set is derived, see Section VII, page 558, for the special case of independently constrained noise signals, the case considered in this paper. While of very significant theoretical value, the results in [32] were not developed to the point of yielding an algorithm for uncertainty set propagation.

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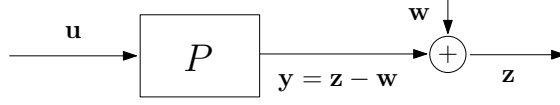


Figure 1: Primal estimation system

In this paper we build on the ideas in [32], particularly that of support function evolution. We use linear programming rather than conjugate functions as our basic tool, and employ the familiar complementary slackness conditions relating primal and dual variables to prove Theorems 9 and 10, and from these results the algorithm is constructed.

2 Basic Setup

The system to be investigated is shown in Fig. 1. The plant P , a linear, time-invariant, causal discrete-time, m^{th} order scalar system, is assumed known. There are two sources of uncertainty, an input noise disturbance $(u_k)_{k=1}^\infty = \mathbf{u}$, and output measurement noise $(w_k)_{k=1}^\infty = \mathbf{w}$. The plant output is $(y_k)_{k=1}^\infty = \mathbf{y}$, and the measurement at time k is $z_k = y_k + w_k$. The initial state, at time $k = 0$, is assumed to be known exactly, but nothing is known about the uncertainties except that they satisfy $|u_k| \leq 1$ and $|w_k| \leq 1$. The system depicted in Fig. 1 will be referred to as the estimation or primal system.

Given an initial state \mathbf{x}_0 , the measurement history z_1, \dots, z_k , and the plant dynamics, we seek the uncertainty set at time k , denoted S_k ; it is the set of possible states consistent with the measurements up to and including z_k , and is easily seen to be a closed, convex polytope.

2.1 Notation

Given a vector $\mathbf{y} = (y_1, y_2, \dots)$ and any $s \in \mathbb{N}^+$, $t \in \mathbb{N}^+$ satisfying $s < t$, we denote $(y_s, y_{s+1}, \dots, y_t)$ by $y_{s:t}$. In matrix equations vectors are by default column vectors, so for example $y_{s:t}$ occurring in a matrix equation would be a column vector, and $y'_{s:t}$ is a row vector, where $'$ denotes transpose. The names of incidence vectors and matrices, whose components are zeros or ones, will start with the letter \mathcal{I} . The λ -transform (generating function) of an arbitrary sequence $\mathbf{y} = (y_k)_{k=1}^\infty$ is defined to be $\hat{\mathbf{y}}(\lambda) := \sum_{k=1}^\infty y_k \lambda^{k-1}$. Real Euclidean space of dimension m is denoted \mathbb{R}^m , where m is the order (McMillan degree) of the plant P . States of the plant P are represented by vectors, or points, in \mathbb{R}^m . Let $\mathbf{d} = d_{1:m+1} = (d_1, \dots, d_{m+1})$ and $\mathbf{n} = n_{1:m+1} = (n_1, \dots, n_{m+1})$, be real vectors, where $\hat{\mathbf{n}}(\lambda)$ and $\hat{\mathbf{d}}(\lambda)$ are the numerator and denominator of the transfer function representation of the plant P . Denote by D and N the infinite, banded, lower-triangular Toeplitz matrices whose first columns are \mathbf{d} and \mathbf{n} , respectively. Define the following submatrices of D .

$$D_L := \begin{bmatrix} d_1 & 0 & \dots & 0 \\ d_2 & d_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ d_m & \dots & d_2 & d_1 \end{bmatrix}$$

$$D_U := \begin{bmatrix} d_{m+1} & d_m & \dots & d_2 \\ 0 & d_{m+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_m \\ 0 & \dots & 0 & d_{m+1} \end{bmatrix}.$$

The matrices N_L and N_U are defined similarly. Generalizing D_L , for any $k > 0$, the $k \times k$ upper left hand corner submatrix of D is denoted $D_{k \times k}$. The matrix $N_{k \times k}$ is defined similarly. Thus $D_{m \times m} = D_L$ and $N_{m \times m} = N_L$.

The Toeplitz Bezoutian matrix of \mathbf{n} and \mathbf{d} is defined as $B_T := D_L N_U - N_L D_U$.

One form of the Gohberg-Semencul formulas [8, 10] states

$$B_T = N_U D_L - D_U N_L, \tag{1}$$

and this will be needed in the proof of Theorem 12, which underpins all of our results. The first row of B_T plays an important role and will be denoted by C , so $C := d_1 [n_{m+1}, \dots, n_2] - n_1 [d_{m+1}, \dots, d_2]$.

The inverse of B_T exists if the polynomials $\hat{\mathbf{n}}(\lambda)$ and $\hat{\mathbf{d}}(\lambda)$ are coprime, and B_T^{-1} denotes the inverse of B_T . See [11] for properties of Bezoutians.

2.2 Transfer function description

The plant for the estimation system has the transfer function representation $P(\lambda) = \hat{\mathbf{n}}(\lambda)/\hat{\mathbf{d}}(\lambda)$ where

$$\begin{aligned}\hat{\mathbf{n}}(\lambda) &= n_1 + n_2\lambda + n_3\lambda^2 + \dots + n_{m+1}\lambda^m \\ \hat{\mathbf{d}}(\lambda) &= d_1 + d_2\lambda + d_3\lambda^2 + \dots + d_{m+1}\lambda^m,\end{aligned}$$

$m \geq 1$ is an integer, $\hat{\mathbf{n}}(\lambda)$ and $\hat{\mathbf{d}}(\lambda)$ are assumed to be coprime polynomials with real coefficients, and it is assumed that both the plant $P(\lambda)$ and the plant $P^*(\lambda)$ for the dual system, defined below, are causal, implying $d_1 \neq 0$ and $d_{m+1} \neq 0$. Without loss of generality we take $d_1 = 1$.

Assuming zero initial conditions, \mathbf{y} and \mathbf{u} are related by $\hat{\mathbf{d}}(\lambda)\hat{\mathbf{y}}(\lambda) = \hat{\mathbf{n}}(\lambda)\hat{\mathbf{u}}(\lambda)$.

2.3 State-space representations

A state-space description of the system depicted in Fig. 1 is

$$\mathbf{x}_k = A\mathbf{x}_{k-1} + Bu_k \quad (2)$$

$$y_k = C\mathbf{x}_{k-1} + \tilde{D}u_k \quad (3)$$

$$z_k = y_k + w_k$$

where

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m-1} \\ -d_{m+1} & \dots & -d_2 \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \quad (4a)$$

$$C = [n_{m+1}, \dots, n_2] - n_1 [d_{m+1}, \dots, d_2], \quad \tilde{D} = n_1; \quad (4b)$$

and

$$\mathbf{x}_k := \begin{cases} B_T^{-1} [D_L y_{k+1:k+m} - N_L u_{k+1:k+m}] & \text{for } k \geq 0 \\ B_T^{-1} [-D_U y_{k-m+1:k} + N_U u_{k-m+1:k}] & \text{for } k \geq m, \end{cases} \quad (5)$$

where \mathbf{x}_k is the state at time k for the sequence pair $(\mathbf{y}, \mathbf{u}) = ((y_j)_{j=1}^\infty, (u_j)_{j=1}^\infty)$. The explicit notation $\mathbf{x}_k(\mathbf{y}, \mathbf{u})$ will sometimes be used for clarity. Note that $\mathbf{x}_k(\mathbf{y}, \mathbf{u}) = A\mathbf{x}_{k-1}(\mathbf{y}, \mathbf{u}) + By_k$ for any input and output sequences (\mathbf{y}, \mathbf{u}) satisfying (2) and (3).

In (4) \mathbf{I}_{m-1} denotes the $m-1$ dimensional identity matrix, and $\mathbf{0}$ denotes a column vector of zeros of length $m-1$.

In (2) the disturbance affects the current state. It is more common to model the disturbance as affecting the future state, but we follow Witsenhausen [32] as it seems natural for our application.

If $n_1 \neq 0$ it follows from (2) and (3) that

$$\begin{aligned}\mathbf{x}_k &= (A - BC/\tilde{D})\mathbf{x}_{k-1} + (B/\tilde{D})y_k \\ &=: \bar{A}\mathbf{x}_{k-1} + \bar{B}y_k\end{aligned} \quad (6)$$

where, using (4),

$$\bar{A} = \begin{bmatrix} 0 & \mathbf{I}_{m-1} \\ -n_{m+1}/n_1 & \dots & -n_2/n_1 \end{bmatrix}, \quad \bar{B} = \frac{1}{n_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (7)$$

There is a system closely related to the estimation system that we refer to as the dual system. The dual system input and output sequences are $(y_j^*)_{j=1}^\infty$ and $(u_j^*)_{j=1}^\infty$, and the dual plant, denoted P^* , has the transfer function representation

$$P^*(\lambda) = -\frac{\tilde{\mathbf{n}}(\lambda)}{\tilde{\mathbf{d}}(\lambda)} \quad (8)$$

where $\tilde{\mathbf{n}} = (n_{m+1}, \dots, n_1)$ and $\tilde{\mathbf{d}} = (d_{m+1}, \dots, d_2, 1)$. See Fig. 2. A minimal state-space realization of the dual system is

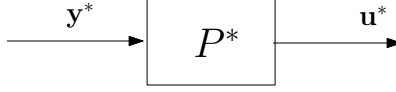


Figure 2: Dual system.

$$\mathbf{x}_k^* = A^* \mathbf{x}_{k-1}^* + B^* y_k^* \quad (9)$$

$$u_k^* = C^* \mathbf{x}_{k-1}^* + \tilde{D}^* y_k^* \quad (10)$$

$$A^* = \begin{bmatrix} -d_m/d_{m+1} & \mathbf{I}_{m-1} \\ \vdots & \\ -1/d_{m+1} & \mathbf{0} \end{bmatrix}, \quad (11)$$

$$B^* = \begin{bmatrix} n_m \\ \vdots \\ n_1 \end{bmatrix} - \begin{bmatrix} d_m \\ \vdots \\ 1 \end{bmatrix} \frac{n_{m+1}}{d_{m+1}}, \quad (12)$$

$$C^* = \begin{bmatrix} -1/d_{m+1} & 0 & \dots & 0 \end{bmatrix}, \quad \tilde{D}^* = \frac{-n_{m+1}}{d_{m+1}}; \quad (13)$$

and

$$\mathbf{x}_k^* := \begin{cases} -N_{\text{U}}^T y_{k+1:k+m}^* - D_{\text{U}}^T u_{k+1:k+m}^* & \text{for } k \geq 0 \\ N_{\text{L}}^T y_{k-m+1:k}^* + D_{\text{L}}^T u_{k-m+1:k}^* & \text{for } k \geq m \end{cases}, \quad (14)$$

where \mathbf{x}_k^* is the dual state at time k for the sequence pair $(\mathbf{y}^*, \mathbf{u}^*) = ((y_j^*)_{j=1}^\infty, (u_j^*)_{j=1}^\infty)$. Then $\mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*) = A^* \mathbf{x}_{k-1}^*(\mathbf{y}^*, \mathbf{u}^*) + B^* y_k^*$ for any input and output sequences $(\mathbf{y}^*, \mathbf{u}^*)$ satisfying (9) and (10).

If $n_{m+1} \neq 0$ it follows from (9) and (10) that

$$\begin{aligned} \mathbf{x}_k^* &= (A^* - B^* C^* / \tilde{D}^*) \mathbf{x}_{k-1}^* + (B^* / \tilde{D}^*) u_k^* \\ &=: \bar{A}^* \mathbf{x}_{k-1}^* + \bar{B}^* u_k^* \end{aligned} \quad (15)$$

where, using (11), (12) and (13)

$$\bar{A}^* = \begin{bmatrix} -n_m/n_{m+1} & I_{m-1} \\ \vdots & \\ -n_1/n_{m+1} & 0 \end{bmatrix}, \quad \bar{B}^* = \begin{bmatrix} d_m \\ \vdots \\ 1 \end{bmatrix} - \frac{d_{m+1}}{n_{m+1}} \begin{bmatrix} n_m \\ \vdots \\ n_1 \end{bmatrix}. \quad (16)$$

The primal and dual state-space representations are in principle well known [13,16,23]. They are given explicitly in [18].

3 Polytopes

The primal and dual states defined in the previous Section will be interpreted in terms of the geometry of the polytopic uncertainty set S_k , so in this Section we introduce notation and briefly summarise the relevant theory of convex polytopes. For more information and background, including the definition of a polytope, see for example [15], [1] or [33]. Let S denote a closed, convex polytope.

The *support function* of S in \mathbb{R}^m is

$$h_S(\mathbf{f}) = \max_{\mathbf{x} \in S} \langle \mathbf{f}, \mathbf{x} \rangle,$$

where $\mathbf{f} \in \mathbb{R}^m$.

When $\mathbf{f} \neq \mathbf{0}$ the set

$$H_S(\mathbf{f}) := \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{f}, \mathbf{x} \rangle = h_S(\mathbf{f})\}$$

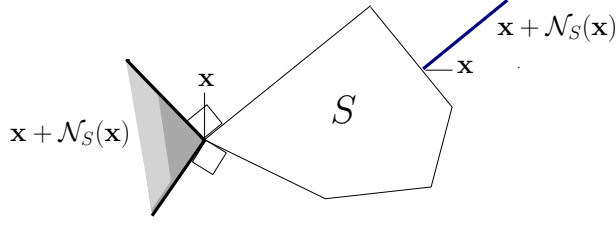


Figure 3: Normal cones to the polytope S

is the supporting hyperplane of S with direction (outer normal vector) \mathbf{f} . If $\mathbf{f} = \mathbf{0}$ then $H_S(\mathbf{f}) = \mathbb{R}^m$.

The intersection of S with a supporting hyperplane is called a *face* of S , and a face of dimension $m - 1$ is a *facet* of S . A face of dimension $m - 2$ is called a *ridge*, and the faces of dimensions 0 and 1 are termed *vertices* and *edges*, respectively. The sets of vertices and boundary points of S are denoted $\text{vert}(S)$ and ∂S . The interior of S is denoted $\text{int}(S)$, and the relative interior of a face F of S is denoted $\text{relint}(F)$.

The *normal cone* at a boundary point \mathbf{x} is the set

$$\{\mathbf{f} \in \mathbb{R}^m : \langle \mathbf{f}, \mathbf{x} \rangle = h_S(\mathbf{f})\}$$

and is denoted $\mathcal{N}_S(\mathbf{x})$. It is generated by the outward normals to the facets that form the polytope at \mathbf{x} , that is

$$\mathcal{N}_S(\mathbf{x}) = \{\lambda_1 \mathbf{f}_1 + \dots + \lambda_n \mathbf{f}_n : \lambda_1, \dots, \lambda_n \geq 0\},$$

where $\mathbf{f}_1, \dots, \mathbf{f}_n$ are the directions of the facets containing \mathbf{x} . Thus $\mathcal{N}_S(\mathbf{x})$ contains the directions of all hyperplanes which touch S at \mathbf{x} . See Fig. 3. If $\mathbf{x} \in \text{int}(S)$, then $\mathcal{N}_S(\mathbf{x}) := \{\mathbf{0}\}$. By definition the directions of facets, and of the hyperplanes that contain facets, point outwards from the polytope. A direction is a non-zero vector.

A useful fact is that \mathbf{x} is in the relative interior of a facet if and only if $\mathcal{N}_S(\mathbf{x})$ contains, in addition to the obligatory zero vector, a non-zero vector that is unique up to multiplication by a non-negative scalar. We will often write *the* direction to denote any such representative of $\mathcal{N}_S(\mathbf{x})$. This situation is illustrated by the normal cone on the right in Fig. 3.

In Section 6.2 the dual state $\mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*)$ will be interpreted as a direction vector \mathbf{f} in the normal cone of the primal state $\mathbf{x}_k(\mathbf{y}, \mathbf{u}) \in S_k$. The symbol \mathbf{f} will be used to denote direction vectors when the geometric viewpoint is being emphasised, while $\mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*)$ will be used to denote the same vector from the system theoretic, algebraic point of view. To reduce notational clutter we shall sometimes drop the subscript k , and indicate the previous time instant with the superscript $-$. For example, \mathbf{f}^- will denote $\mathbf{x}_{k-1}^*(\mathbf{y}^*, \mathbf{u}^*)$. The alternative notations are summarised below:

$$\begin{aligned} \mathbf{f} &\leftrightarrow \mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*) \\ \mathbf{f}^- &\leftrightarrow \mathbf{x}_{k-1}^*(\mathbf{y}^*, \mathbf{u}^*) \\ \mathbf{x} &\leftrightarrow \mathbf{x}_k(\mathbf{y}, \mathbf{u}) \\ \mathbf{x}^- &\leftrightarrow \mathbf{x}_{k-1}(\mathbf{y}, \mathbf{u}) \end{aligned}$$

where the sequence pairs (\mathbf{y}, \mathbf{u}) and $(\mathbf{y}^*, \mathbf{u}^*)$ are defined in Section 2.3.

4 State propagation

The primal system at time zero is in the state \mathbf{x}_0 , so $D_L y_{1:m} - N_L u_{1:m} = B_T \mathbf{x}_0$. After $k \geq 2m$ measurements have been processed $y_{1:k}$ and $u_{1:k}$ are related by

$$D_{k \times k} y_{1:k} - N_{k \times k} u_{1:k} = \begin{bmatrix} B_T \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix}. \quad (17)$$

Following Witsenhausen, [32], S_k is given recursively in terms of S_{k-1} and the new observation z_k by

$$S_k = \left\{ \mathbf{x}_k : \begin{array}{l} \mathbf{x}_{k-1} \in S_{k-1}, \mathbf{x}_k = A\mathbf{x}_{k-1} + B\mathbf{u}_k, \\ y_k = C\mathbf{x}_{k-1} + \tilde{D}\mathbf{u}_k, \\ |u_k| \leq 1, |y_k - z_k| \leq 1. \end{array} \right\} \quad (18)$$

Special notation is now introduced to describe states \mathbf{x}_{k-1} and \mathbf{x}_k related as in (18).

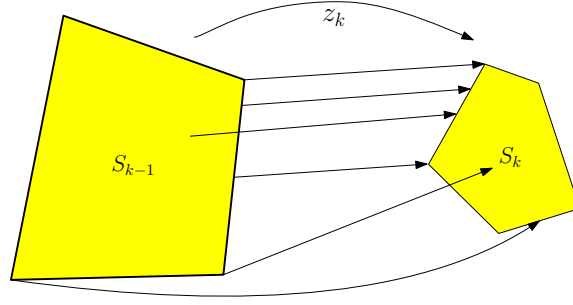


Figure 4: Typical propagations of a state in S_{k-1} to its successor in S_k . From top to bottom:
vertex to vertex;
non-vertex boundary point to non-vertex boundary point;
interior point to non-vertex boundary point;
non-vertex boundary point to vertex;
vertex to interior; and
vertex to non-vertex boundary point.

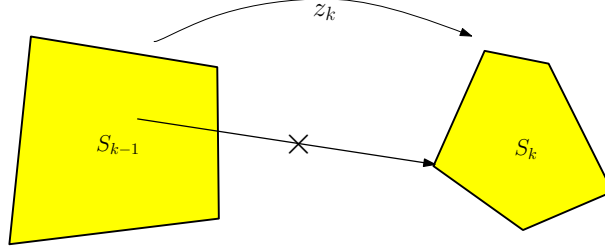


Figure 5: Interior point to vertex is not possible.

Definition 1. The state $\mathbf{x}_{k-1} \in S_{k-1}$ is said to be a precursor of the state \mathbf{x}_k , \mathbf{x}_{k-1} is propagated to \mathbf{x}_k , and \mathbf{x}_k is a successor to \mathbf{x}_{k-1} , if there exists a scalar u_k satisfying (3), $|u_k| \leq 1$, and $|y_k - z_k| \leq 1$.

So S_k is the set of all successors to all states in S_{k-1} , and any precursor of any state $\mathbf{x}_k \in S_k$ is in S_{k-1} . See Fig. 4 for some typical modes of propagation.

We are going to systematically consider the propagation of points in S_{k-1} whose successors lie on the boundary of S_k . It will be shown that, in order to determine all successors to a state $\mathbf{x}_{k-1} \in S_{k-1}$, it is necessary and sufficient to know just z_k , \mathbf{x}_{k-1} , and $\mathcal{N}_{S_{k-1}}(\mathbf{x}_{k-1})$; the rest of S_{k-1} is irrelevant.

The uncertainty sets will be interpreted as feasible sets for specially constructed optimisation problems, and optimal solutions to these programs are often points on the boundaries of the uncertainty sets. We will find the boundary of S_k by locating all precursors of all boundary points of S_k , and then propagating them. There are two possibilities for the location of a precursor of a point on the boundary of S_k : either it is in the interior of S_{k-1} (interior to boundary propagation), or it lies on the boundary of S_{k-1} (boundary to boundary). It also helps to know when points on the boundary of S_{k-1} are propagated to the interior of S_k , because they can be safely ignored. An understanding of all modes of propagation is necessary.

Possible propagations of states in S_{k-1} are shown in Fig. 4. It is also possible for a state on the boundary of S_{k-1} to have two, or even an infinite number of, successors on the boundary of S_k . Another possibility is for a state in S_{k-1} to have no successor at all; the measurement may be inconsistent with a given state, or even inconsistent with every state, in S_{k-1} . In this latter case, S_k is empty.

One mode of propagation that is not possible is interior point to vertex, see Fig. 5. It will be shown that an even stronger property holds: the precursors of vertices of S_k are either vertices of S_{k-1} or are points in the relative interior of edges of S_{k-1} .

State propagation is defined in terms of the inputs and outputs of the plant, and these lie on primal lines, defined next.

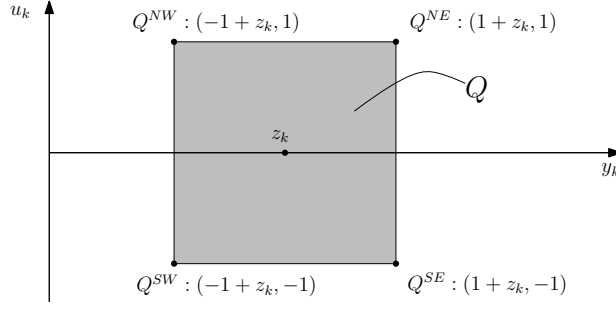


Figure 6: The square Q .

4.1 Definitions of primal and dual lines, and the square Q

Associated with any state $\mathbf{x}^- \in S_{k-1}$ define in the $y_k u_k$ -plane the primal line $L(\mathbf{x}^-)$:

Definition 2.

$$L(\mathbf{x}^-) = \{(y_k, u_k) : y_k - n_1 u_k = C\mathbf{x}^-\}.$$

The input and output of the plant P at time k are constrained to lie on the line $L(\mathbf{x}^-)$ by (3) and (4).

Although not required in this Section, we will need later lines constructed, in similar fashion, from dual states. Associated with any $\mathbf{f}^- \in \mathbb{R}^m$ define in the $y_k^* u_k^*$ -plane the dual line $L^*(\mathbf{f}^-)$:

Definition 3.

$$L^*(\mathbf{f}^-) = \{(y_k^*, u_k^*) : n_{m+1} y_k^* + d_{m+1} u_k^* = -(\mathbf{f}^-)_1\}.$$

The scalars y_k^* and u_k^* , the input and output of the dual plant at time k , are constrained to lie on the line $L^*(\mathbf{f}^-)$ by (10) and (13).

In the $y_k u_k$ -plane define the square Q containing points (y_k, u_k) satisfying $|u_k| \leq 1$ and $|y_k - z_k| \leq 1$, so Q contains the allowable outputs and inputs to the plant at time k .

Definition 4.

$$Q = \{(y_k, u_k) : |u_k| \leq 1 \text{ and } |y_k - z_k| \leq 1\}.$$

Thus Q is the set of points on or inside the square with corners $Q^{NE} : (1 + z_k, 1)$, $Q^{SE} : (1 + z_k, -1)$, $Q^{SW} : (-1 + z_k, -1)$ and $Q^{NW} : (-1 + z_k, 1)$. The square Q has sides of length two and is centred at $(z_k, 0)$. See Fig. 6. The set of points in the interior of the top (bottom, left, right) sides of Q is denoted T (B, L, R), so the boundary of Q is $T \cup B \cup L \cup R \cup Q^{SE} \cup Q^{NE} \cup Q^{SW} \cup Q^{NW}$.

4.2 Does the state $\mathbf{x}^- \in S_{k-1}$ have a successor?

We now consider the question of when a given $\mathbf{x}^- \in S_{k-1}$ has a successor at all, since there will be values of z_k which are inconsistent with the plant being in the state \mathbf{x}^- at time $k - 1$. In accordance with Definitions 1 and 4 this occurs when the line $L(\mathbf{x}^-)$ does not intersect the square Q . The set \widetilde{M} defined next contains the u_k components of the points of intersection (y_k, u_k) of $L(\mathbf{x}^-)$ with Q .

Definition 5. Given $\mathbf{x}^- \in S_{k-1}$ and z_k , the set $\widetilde{M}(\mathbf{x}^-, z_k)$ is

$$\widetilde{M} := \left\{ u_k : \begin{array}{l} |u_k| \leq 1 \text{ and } |y_k - z_k| \leq 1, \\ \text{where } (y_k, u_k) \in L(\mathbf{x}^-). \end{array} \right\}$$

Proposition 6. The state $\mathbf{x}^- \in S_{k-1}$ has a successor $\mathbf{x}_k \in S_k$ if and only if $\widetilde{M}(\mathbf{x}^-, z_k)$ is non-empty. Furthermore, the set of all successors to \mathbf{x}^- is

$$\{A\mathbf{x}^- + Bu_k : u_k \in \widetilde{M}(\mathbf{x}^-, z_k)\}.$$

Proof. By Definition 1, \mathbf{x}^- has a successor if and only if there exists $u_k \in \widetilde{M}(\mathbf{x}^-, z_k)$, in which case $A\mathbf{x}^- + Bu_k$ is a successor to \mathbf{x}^- . \square

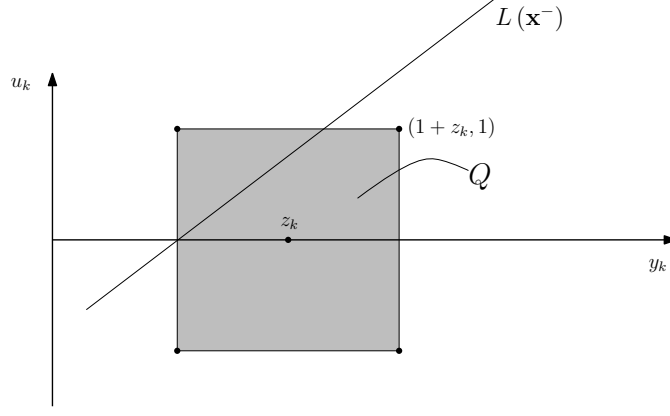


Figure 7: $A\mathbf{x}^- + Bu_k$ is a successor to \mathbf{x}^- for all $u_k \in \widetilde{M}(\mathbf{x}^-, z_k) = [0, 1]$.

This is illustrated in In Fig. 7, where the points of intersection, (y_k, u_k) , of the line $L(\mathbf{x}^-)$ with the square Q satisfy $u_k \in \widetilde{M}(\mathbf{x}^-, z_k) = [0, 1]$.

5 Statement of main results

The important and challenging question, to which we now turn, is how to determine which states in S_{k-1} propagate to the *boundary* of S_k . Our main results answer this question and moreover have a nice geometrical interpretation. Both the statement of the results and their proofs rely on the duality existing between two optimisation programs.

The decision variables for the primal optimisation problem, to be defined in Section 6, are the primal system input and output signals \mathbf{u} and \mathbf{y} . The decision variables for the dual optimisation problem, also to be defined in Section 6, are the dual system input and output signals \mathbf{y}^* and \mathbf{u}^* . A primal state, which is defined in terms of primal input and output signals, is a point in the uncertainty set, and vectors in the normal cone of primal states are interpreted as states of dual input and output signals.

5.1 Alignment between primal and dual signals

At optimality the primal and dual signals will be related through, in linear programming terminology, complementary slackness. The particular form this relationship takes in our setup is termed alignment, and is defined next.

Recall that, at time k , the scalars y_k and u_k are the output and input of the plant, and y_k^* and u_k^* are the input and output of the dual plant.

Definition 7. *The scalar pair (y_k, u_k) is said to be aligned with (y_k^*, u_k^*) if*

$$\begin{aligned} u_k^* > 0 &\implies u_k = 1 \\ u_k^* < 0 &\implies u_k = -1 \\ |u_k| < 1 &\implies u_k^* = 0 \end{aligned} \tag{19}$$

and

$$\begin{aligned} y_k^* > 0 &\implies y_k = 1 + z_k \\ y_k^* < 0 &\implies y_k = -1 + z_k \\ |y_k - z_k| < 1 &\implies y_k^* = 0. \end{aligned} \tag{20}$$

This definition can be extended in a natural way to alignment between pairs of sequences. Thus the vector pair (\mathbf{y}, \mathbf{u}) is aligned with the pair $(\mathbf{y}^*, \mathbf{u}^*)$ if, for all k , (y_k, u_k) is aligned with (y_k^*, u_k^*) .

Alignment between points in the $y_k u_k$ and $y_k^* u_k^*$ -planes can be readily visualised as follows.

By (19) and (20) each point (y_k, u_k) belonging to the top (bottom) side of Q is aligned with every point on the positive (negative) u_k^* axis in the $y_k^* u_k^*$ -plane. See Fig. 8. Similarly, each point (y_k, u_k) belonging to the right (left) side of Q is aligned with every point on the positive (negative) y_k^* axis in the $y_k^* u_k^*$ -plane.

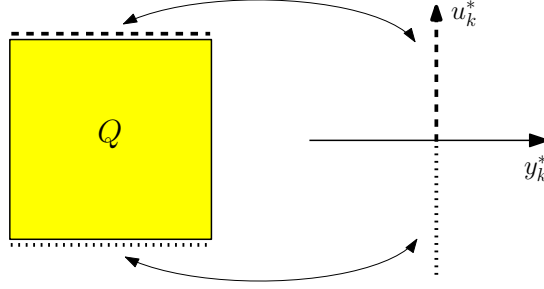


Figure 8: Alignment between the top (bottom) side of Q and the positive (negative) u_k^* axis.

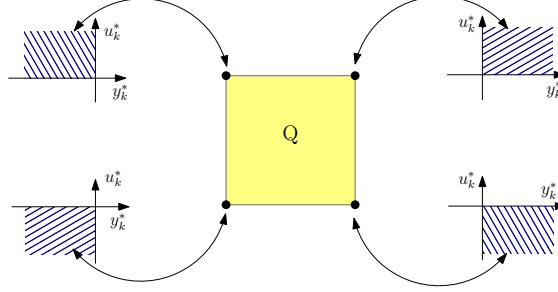


Figure 9: Alignment between the corners of Q and the four quadrants in the $y_k^* u_k^*$ -plane.

The corner $Q^{\text{NE}}(Q^{\text{NW}}, Q^{\text{SW}}, Q^{\text{SE}})$ of Q is aligned with all points in the first (respectively, second, third, fourth) quadrant of the $y_k^* u_k^*$ -plane. See Fig. 9.

Finally, all points in Q are aligned with the origin in the $y_k^* u_k^*$ -plane.

5.2 Recursive procedure

In order to state our main results we need to extend the definition of \widetilde{M} to include aligned primal and dual signals. It will play a central role in the rest of the paper.

Definition 8. Given $\mathbf{x}^- \in S_{k-1}$, $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ and z_k , the set $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is

$$M := \left\{ (u_k, y_k^*) \in \mathbb{R}^2 : \begin{array}{l} |u_k| \leq 1; \\ |y_k - z_k| \leq 1, \text{ where } (y_k, u_k) \in L(\mathbf{x}^-); \\ (y_k, u_k) \text{ is aligned with } (y_k^*, u_k^*), \text{ where} \\ (y_k^*, u_k^*) \in L^*(\mathbf{f}^-). \end{array} \right\}$$

The inequality constraints in Definition 8, which give the u_k components of points of intersection of $L(\mathbf{x}^-)$ with Q , are the same as those in the definition of \widetilde{M} , see Definition 5. Definition 8 further restricts u_k by requiring alignment to be satisfied. Finding M is computationally very simple, requiring merely the intersection of lines in the plane, and checking alignment. For example, in Fig. 10 the line $L(\mathbf{x}^-)$ intersects the top and bottom sides of Q , so $\widetilde{M} = [-1, 1]$, but alignment occurs solely between the intersection of $L(\mathbf{x}^-)$ with the top side of Q , and the intersection of $L^*(\mathbf{f}^-)$ with the positive u_k^* axis. Hence M contains the singleton pair $(1, 0)$.

We now present Theorem 9, the basis of our recursive procedure for determining ∂S_k from S_{k-1} . See Fig. 11 for a geometric depiction of the vectors in Theorem 9 for the important special case where \mathbf{x} and \mathbf{x}^- are both boundary points. The proof relies on Theorem 12 and is discussed in Section 6. Details are in the Appendix.

Theorem 9. Suppose $\mathbf{x}^- \in S_{k-1}$ and $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$. Then $\mathbf{x} = A\mathbf{x}^- + Bu_k \in S_k$ and $\mathbf{f} = A^*\mathbf{f}^- + B^*y_k^* \in \mathcal{N}_{S_k}(\mathbf{x})$ if and only if $(u_k, y_k^*) \in M(\mathbf{x}^-, \mathbf{f}^-, z_k)$.

This result is the “engine” driving the recursive procedure. Often $\mathbf{0} \neq \mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x})$, implying $\mathbf{x} \in \partial S_k$. In a typical application a point on the boundary of S_{k-1} , and the direction of any one of its supporting hyperplanes, are propagated to a point on the boundary of S_k along with the direction of one of its supporting hyperplanes.

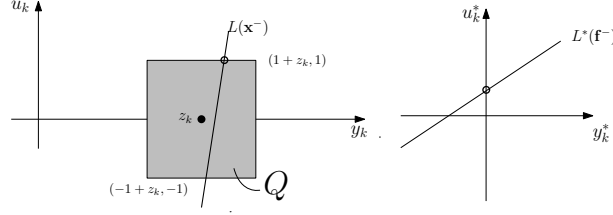


Figure 10: The circled points on the lines $L(\mathbf{x}^-)$ and $L^*(\mathbf{f}^-)$ are aligned, so $M(\mathbf{x}^-, \mathbf{f}^-, z_k) = \{(1, 0)\}$ because at the aligned points $u_k = 1$ and $y_k^* = 0$.

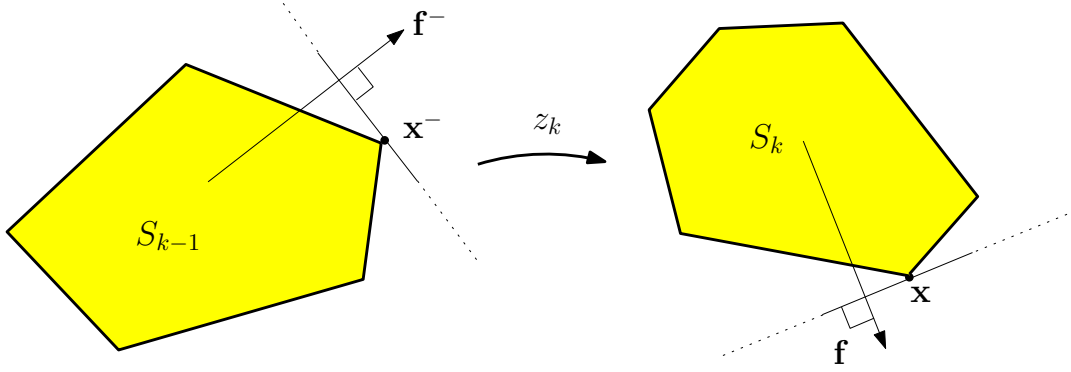


Figure 11: The vectors in Theorem 9. The state $\mathbf{x}^- \in \partial S_{k-1}$ and direction $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ are propagated to $\mathbf{x} \in \partial S_k$ and $\mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x})$ by the measurement z_k .

5.3 Precursors of a given $\mathbf{x} \in S_k$

For given $\mathbf{x}^- \in S_{k-1}$ and $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ Theorem 9 furnishes at least one successor \mathbf{x} to \mathbf{x}^- , and one vector $\mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x})$, if $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is non-empty. It has not yet been shown, however, that for *any* point \mathbf{x} of S_k there exists \mathbf{x}^- and \mathbf{f}^- as in the Theorem statement. In other words, it is not yet clear that *every* point of S_k can be found through the process of applying Theorem 9 to some $\mathbf{x}^- \in S_{k-1}$ and some $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$.

The companion result Theorem 10, given below, shows that indeed any boundary point $\mathbf{x} \in S_k$, and any direction in the normal cone of \mathbf{x} , are attainable from *any* precursor \mathbf{x}^- of \mathbf{x} and *some* direction in the normal cone of \mathbf{x}^- . Together Theorems 9 and 10 provide all the information needed to construct S_k from S_{k-1} .

Theorem 10. *Select any $\mathbf{x} \in S_k$, any $\mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x})$ and any precursor \mathbf{x}^- of \mathbf{x} . There exists $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ and $(u_k, y_k^*) \in M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ for which $\mathbf{x} = A\mathbf{x}^- + Bu_k$ and $\mathbf{f} = A^*\mathbf{f}^- + B^*y_k^*$.*

An outline of the proof is in Section 6.2. Details are in the Appendix.

Having stated our main results, we next provide the dual optimisation framework required for their derivation.

6 The primal estimation program and its dual

In this Section we set up primal and dual optimisation programs. Their decision variables are the input and output signals of the primal and dual systems.

6.1 The primal estimation program $\mathcal{P}_{z_{1:k}}(\mathbf{f})$

At any time $k \geq 2m$, and for any $\mathbf{f} \in \mathbb{R}^m$, consider the program

$$\max_{\mathbf{x} \in S_k} \langle \mathbf{f}, \mathbf{x} \rangle. \quad (21)$$

It has optimal value $h_{S_k}(\mathbf{f})$, the support function of S_k evaluated at \mathbf{f} .

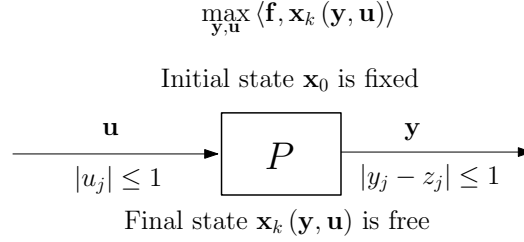


Figure 12: The program $\mathcal{P}_{z_{1:k}}(\mathbf{f})$. If $\mathbf{f} \neq \mathbf{0}$ then optimizing \mathbf{x}_k are points on the boundary of S_k .

Writing out the constraints explicitly in terms of the output and input signals up to time k , $(\mathbf{y}, \mathbf{u}) := (y_{1:k}, u_{1:k})$, by (17) the program (21) can be equivalently expressed as:

$$\begin{aligned} \mathcal{P}_{z_{1:k}}(\mathbf{f}) : \quad & \max_{\mathbf{y}, \mathbf{u}} \langle \mathbf{f}, \mathbf{x}_k(\mathbf{y}, \mathbf{u}) \rangle \\ & \text{subject to} \\ & D_{k \times k} \mathbf{y}_{1:k} - N_{k \times k} \mathbf{u}_{1:k} = \begin{bmatrix} B_T \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix}. \end{aligned}$$

The matrices $D_{k \times k}$ and $N_{k \times k}$ are defined in Section 2.1. From now on we will always assume that the constraints for the primal program are consistent, which is equivalent to saying that all of the uncertainty sets up to time k are non-empty.

The decision variables for the program $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ can be thought of as either the outputs and inputs of the primal system up to time k or, as in (21), the states of S_k . By (5) the state of the primal system at time k is $\mathbf{x}_k(\mathbf{y}, \mathbf{u})$. See Fig 12.

The following Proposition is an easy consequence of the definitions.

Proposition 11. *For any $\mathbf{x} \in S_k$ there exists (\mathbf{y}, \mathbf{u}) feasible for $\mathcal{P}_{z_{1:k}}(\cdot)$ for which $\mathbf{x} = \mathbf{x}_k(\mathbf{y}, \mathbf{u})$. For any such (\mathbf{y}, \mathbf{u}) , and any $\mathbf{f} \in \mathbb{R}^m$, there holds*

$$\begin{aligned} (\mathbf{y}, \mathbf{u}) \in \arg \max \mathcal{P}_{z_{1:k}}(\mathbf{f}) & \Leftrightarrow h_{S_k}(\mathbf{f}) = \langle \mathbf{f}, \mathbf{x} \rangle \\ & \Leftrightarrow \mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x}). \end{aligned}$$

Note that if (\mathbf{y}, \mathbf{u}) is feasible for $\mathcal{P}_{z_{1:k}}(\cdot)$ then, for $i = 1, \dots, k$, $\mathbf{x}_i = A\mathbf{x}_{i-1} + B\mathbf{u}_i \in S_i$ and $y_i = C\mathbf{x}_{i-1} + \tilde{D}u_i$.

6.2 The dual program $\mathcal{D}_{z_{1:k}}(\mathbf{f})$

Let $\mathbf{f} \in \mathbb{R}^m$ be given. The proof of Theorem 9 is based on the duality existing between $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ and the program introduced next, denoted $\mathcal{D}_{z_{1:k}}(\mathbf{f})$.

$$\begin{aligned} \mathcal{D}_{z_{1:k}}(\mathbf{f}) : \quad & \min_{\mathbf{y}^*, \mathbf{u}^*} \{ \|\mathbf{y}^*\|_1 + \|\mathbf{u}^*\|_1 + \langle \mathbf{y}^*, \mathbf{z} \rangle + \langle \mathbf{x}_0^*(\mathbf{y}^*, \mathbf{u}^*), \mathbf{x}_0 \rangle \} \\ & \text{subject to} \\ & N_{k \times k}^T \mathbf{y}_{1:k}^* + D_{k \times k}^T \mathbf{u}_{1:k}^* = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}. \end{aligned}$$

The decision variables, $(\mathbf{y}^*, \mathbf{u}^*) := (y_{1:k}^*, u_{1:k}^*)$, are the inputs and outputs up to time k of the dual system. See Fig 13. The matrix $D_{k \times k}^T (N_{k \times k}^T)$ denotes the transpose of $D_{k \times k}$ ($N_{k \times k}$), so the bottom right hand corner m by m submatrix of $D_{k \times k}^T (N_{k \times k}^T)$ is the transpose of $D_L (N_L)$. Thus, by (14), the last m of the constraint equations state that the decision variables are constrained by $\mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*) = \mathbf{f}$.

A formal statement of the duality relating $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ to $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ is now presented. It is the basis for all new results in this paper. Amongst other things, it justifies the interpretation of $\mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*)$ as the direction vector \mathbf{f} , the argument of the support function of S_k , see (21). It is worth pointing out that the structurally elegant form manifest in this result is not apparent in a routine application of duality to $\mathcal{P}_{z_{1:k}}(\mathbf{f})$. Observe, for example, that for the program $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ the initial state is free, and the terminal state is fixed, at \mathbf{f} . For the program $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ the initial state is fixed, at \mathbf{x}_0 , and the terminal state is free. Some finesse is required in the construction of the dual

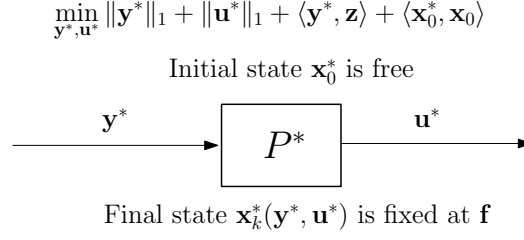


Figure 13: The program $\mathcal{D}_{z_{1:k}}(\mathbf{f})$

variables and the dual cost function. Any dual of $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ will be equivalent to $\mathcal{D}_{z_{1:k}}(\mathbf{f})$, but it is only $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ that furnishes a proof of Theorem 9.

Theorem 12. *Suppose the set S_k is non-empty. Then the optimal values of $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ and $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ are finite and equal. Furthermore, if (\mathbf{y}, \mathbf{u}) and $(\mathbf{y}^*, \mathbf{u}^*)$ are feasible for $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ and $\mathcal{D}_{z_{1:k}}(\mathbf{f})$, respectively, then a necessary and sufficient condition that they both be optimal is that they be aligned.*

Proof. The proof in outline follows standard linear programming arguments, although some non-routine manipulations involving the Gohberg-Semencul formula (1) are also required. Details are in the Appendix. \square

Remark 13. *It can be shown that, under the standing assumption that $\hat{\mathbf{n}}$ and $\hat{\mathbf{d}}$ are coprime, $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ always has a feasible solution, and has unbounded negative cost if S_k is empty.*

Armed with Theorem 12 we can now provide proofs of Theorems 9 and 10. Suppose we are given $(y_{1:k-1}, u_{1:k-1}) \in \arg \max \mathcal{P}_{z_{1:k-1}}(\mathbf{f}^-)$ and $(y_{1:k-1}^*, u_{1:k-1}^*) \in \arg \min \mathcal{D}_{z_{1:k-1}}(\mathbf{f}^-)$. The proof of Theorem 9 uses Theorem 12 to answer the key question: What are the extensions (if any) of $(y_{1:k-1}, u_{1:k-1})$ to $(y_{1:k}, u_{1:k}) =: (\mathbf{y}, \mathbf{u})$, and $(y_{1:k-1}^*, u_{1:k-1}^*)$ to $(y_{1:k}^*, u_{1:k}^*) =: (\mathbf{y}^*, \mathbf{u}^*)$, satisfying $(\mathbf{y}, \mathbf{u}) \in \arg \max \mathcal{P}_{z_{1:k}}(\mathbf{x}_k^*(\mathbf{y}^*, \mathbf{u}^*))$? See the Appendix for details.

The proof of Theorem 10, on the other hand, starts with optimal sequences of length k , and uses a dynamic programming style argument in conjunction with Theorems 9 and 12 to derive properties of truncations of length $k - 1$. Details are in the Appendix.

As a first application of Theorems 9 and 10 in the next section we find when points on the boundary of S_{k-1} propagate to the interior of S_k .

6.3 Propagation from boundary to interior

The proof of the following Proposition, which relies on Theorems 9 and 10, is in the Appendix.

Proposition 14. *Select any $\mathbf{x}^- \in \partial S_{k-1}$. All successors to \mathbf{x}^- belong to the interior of S_k if and only if $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is empty for all $\mathbf{0} \neq \mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$.*

An example where $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is empty is shown in Fig. 14; there is no alignment between any point on $L(\mathbf{x}^-)$ with any point on $L^*(\mathbf{f}^-)$, yet $L(\mathbf{x}^-)$ does intersect the square Q . This is the “boundary to interior” case depicted in Fig. 4.

We now have the tools to determine where any state is propagated. In the remainder of the paper we use them to construct an algorithm which updates the whole polytope.

7 The facets and vertices of S_k

In the rest of the paper we show how to use the propagation of states and their normal cones to update the uncertainty set for plants of any order satisfying Conditions 20, 21 and 23 given below. This restriction is due to space limitations. We present a simple example and Matlab code, `uncertaintyset.m`, available on the link below¹, which gives exact solutions for rational data. Although the details are somewhat involved, the procedure is computationally effective because, apart from preliminary sorting, it is only logical operations on incidence vectors and matrices that are needed to construct the updated incidence matrix.

We need notation to describe four typical ways a state might be propagated. The four mappings g^T , g^B , g^L , and g^R , where the superscripts indicate the top, bottom, left and right sides of Q , all map a state in S_{k-1} to a

¹Go to <https://mathworks.com/matlabcentral/fileexchange>

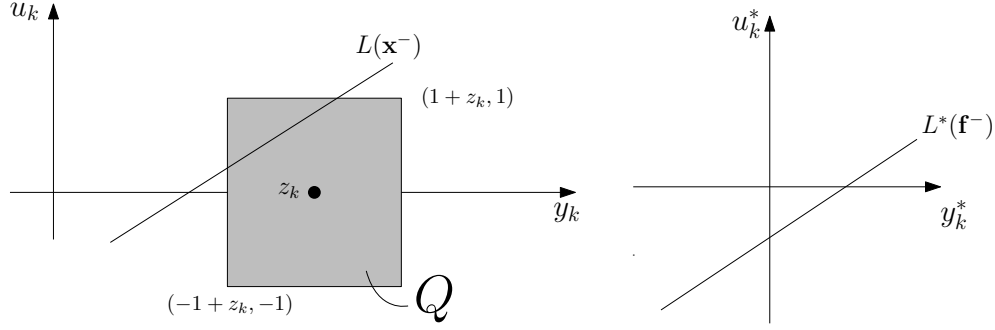


Figure 14: The set $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is empty.

state of the plant at time k . On the top (bottom) side of Q , $u_k = 1$ ($u_k = -1$), and (2) is used to propagate \mathbf{x}^- ; for the left (right) side of Q , $y_k = z_k - 1$ ($y_k = z_k + 1$) and (6) is used to propagate \mathbf{x}^- .

Definition 15. For any $\mathbf{x}^- \in S_{k-1}$ put

$$\begin{aligned} g^T(\mathbf{x}^-) &= A\mathbf{x}^- + B \\ g^B(\mathbf{x}^-) &= A\mathbf{x}^- - B \\ g^L(\mathbf{x}^-) &= \bar{A}\mathbf{x}^- + \bar{B}(z_k - 1) \\ g^R(\mathbf{x}^-) &= \bar{A}\mathbf{x}^- + \bar{B}(z_k + 1). \end{aligned} \quad (22)$$

Definition 16. A point $\mathbf{x}^- \in S_{k-1}$ is said to propagate up (resp. down, left, right) if $g^T(\mathbf{x}^-)$ ($g^B(\mathbf{x}^-)$, $g^L(\mathbf{x}^-)$, $g^R(\mathbf{x}^-)$) lies on the boundary of S_k .

These propagations occur when the minimum or maximum allowable value of the input (± 1) is applied to the plant, or the minimum or maximum allowable value, $z_k \pm 1$, occurs at the output of the plant, and furthermore the resulting state of the plant lies on the boundary of S_k . Theorem 9 tells us, given \mathbf{x}^- and $\mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$, whether or not \mathbf{x}^- propagates up, down, left or right. For example, in Fig. 10, \mathbf{x}^- propagates up to $g^T(\mathbf{x}^-)$.

The significance of the maps in Definition 16 is made clear by the following Proposition. The proof is in the Appendix.

Proposition 17. Suppose \mathbf{x} is a vertex of S_k with precursor \mathbf{x}^- . Then \mathbf{x} must equal one of $g^T(\mathbf{x}^-)$, $g^B(\mathbf{x}^-)$, $g^L(\mathbf{x}^-)$ or $g^R(\mathbf{x}^-)$.

The matrices A and \bar{A} are invertible so the four maps defined in (22) are one-to-one and affine. Since a one-to-one affine map of a polytope produces an affinely isomorphic polytope, we have the following result, stated here for future reference.

Proposition 18. If $r \leq m$, and S is an r -dimensional polytope, then $g^T(S)$, $g^B(S)$, $g^L(S)$ and $g^R(S)$ are r -dimensional polytopes. Furthermore, \mathbf{v}^- is a vertex of S if and only if $g^T(\mathbf{v}^-)$ (resp $g^B(\mathbf{v}^-)$, $g^L(\mathbf{v}^-)$, $g^R(\mathbf{v}^-)$) is a vertex of $g^T(S)$ (resp $g^B(S)$, $g^L(S)$, $g^R(S)$).

7.1 Hyperplanes and half spaces associated with the square Q

Those states \mathbf{x}^- for which the primal line $L(\mathbf{x}^-)$ intersects a corner of Q are important, so we introduce notation to describe this situation.

Definition 19. $s^{\text{NW}} = z_k - 1 - n_1$, $s^{\text{NE}} = z_k + 1 - n_1$, $s^{\text{SW}} = z_k - 1 + n_1$ and $s^{\text{SE}} = z_k + 1 + n_1$.

Then, for example, $L(\mathbf{x}^-)$ intersects the corner point Q^{NW} in the $y_k u_k$ plane if and only if $C\mathbf{x}^- = s^{\text{NW}}$.

The details of algorithm implementation depend to some extent on the signs of the leading and trailing coefficients of the polynomials $\hat{\mathbf{d}}(\lambda)$ and $\hat{\mathbf{n}}(\lambda)$, and we give a detailed description only for the case where the primal and dual lines both have positive slope, that is $n_1 > 0$ and $n_{m+1}d_{m+1} < 0$. It can be readily verified that if $n_1 > 0$ then $s^{\text{NW}} < s^{\text{SW}} < s^{\text{SE}}$ and $s^{\text{NW}} < s^{\text{NE}} < s^{\text{SE}}$.

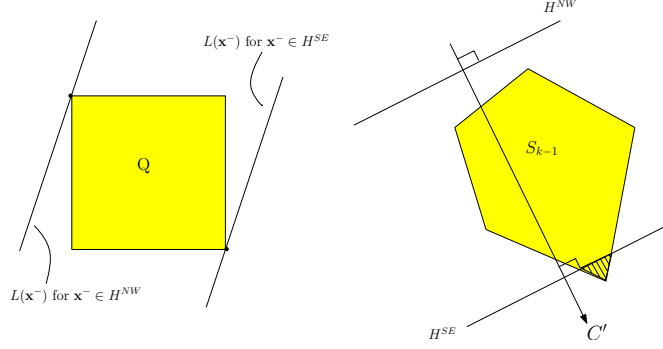


Figure 15: The primal lines associated with the hyperplanes H^{NW} and H^{SE} . The shaded area is removed from S_{k-1} .

Condition 20. $n_1 > 0$ and $n_{m+1}d_{m+1} < 0$.

A simplifying assumption we make is:

Condition 21. $\mathcal{IV}^{NE} = \mathcal{IV}^{SW} = \mathbf{0}$.

If Condition 21 is not satisfied then there are two options. A small perturbation can be applied to z_k , which will have the effect of moving the square Q slightly, so the primal line of the offending vertex will no longer pass through a corner of Q . Alternatively, a few lines of code can be added to take care of this special situation.

We define four hyperplanes and eight half planes in \mathbb{R}^m associated with the corners of Q .

Definition 22.

$$\begin{aligned} H^{NW} &:= \{\mathbf{x}^- : C\mathbf{x}^- = s^{NW}\}; H^{NW+} := \{\mathbf{x}^- : C\mathbf{x}^- \geq s^{NW}\}; H^{NW-} := \{\mathbf{x}^- : C\mathbf{x}^- \leq s^{NW}\} \\ H^{SW} &:= \{\mathbf{x}^- : C\mathbf{x}^- = s^{SW}\}; H^{SW+} := \{\mathbf{x}^- : C\mathbf{x}^- \geq s^{SW}\}; H^{SW-} := \{\mathbf{x}^- : C\mathbf{x}^- \leq s^{SW}\} \\ H^{NE} &:= \{\mathbf{x}^- : C\mathbf{x}^- = s^{NE}\}; H^{NE+} := \{\mathbf{x}^- : C\mathbf{x}^- \geq s^{NE}\}; H^{NE-} := \{\mathbf{x}^- : C\mathbf{x}^- \leq s^{NE}\} \\ H^{SE} &:= \{\mathbf{x}^- : C\mathbf{x}^- = s^{SE}\}; H^{SE+} := \{\mathbf{x}^- : C\mathbf{x}^- \geq s^{SE}\}; H^{SE-} := \{\mathbf{x}^- : C\mathbf{x}^- \leq s^{SE}\} \end{aligned}$$

For example, for any $\mathbf{x}^- \in H^{NW}$ the line $L(\mathbf{x}^-)$ intersects Q^{NW} , and similarly for the other corners. See Figs. 15 and 16.

The intersections (if any) of the four hyperplanes defined in Definition 22 with S_{k-1} will be needed, so we summarise notation here. The vertex matrix, \mathbf{V}^{HNW} , has columns, in an arbitrary but fixed order, containing the coordinates of the vertices of $H^{NW} \cap S_{k-1}$. Thus the columns of \mathbf{V}^{HNW} are $(\mathbf{v}_1^{-HNW}, \dots, \mathbf{v}_{n_v^{HNW}}^{-HNW})$, where $\mathbf{v}_j^{-HNW} \in \text{vert}(H^{NW} \cap S_{k-1})$ and n_v^{HNW} is the number of vertices in $\text{vert}(H^{NW} \cap S_{k-1})$. The vectors \mathbf{v}_j^{-HNW} are the points of intersection of the hyperplane H^{NW} with the edges of S_{k-1} . See Figs. 15 and 16.

The vertex matrices \mathbf{V}^{HSW} , \mathbf{V}^{HNE} and \mathbf{V}^{HSE} , are defined similarly to \mathbf{V}^{HNW} . Also their column dimensions are n_v^{HSW} , n_v^{HNE} and n_v^{HSE} , in analogy with the use of n_v^{HNW} to denote the number of columns in \mathbf{V}^{HNW} . Depending on the value of z_k , these vertex matrices may be empty. Vertices \mathbf{v}_j^{-HSE} and \mathbf{v}_j^{-HNW} are shown in Fig. 16.

States with no successors

In the right side of Fig. 15 the shaded region is $H^{SE+} \cap S_{k-1}$. The states in the shaded region have no successors and the first step of the algorithm is to remove all such states from S_{k-1} . In general, the set of states in S_{k-1} with at least one successor is $S_{k-1} \cap H^{NW+} \cap H^{SE-}$ so, given any z_k , propagating $S_{k-1} \cap H^{NW+} \cap H^{SE-}$ is the same as propagating S_{k-1} . States which do not propagate are simply removed from S_{k-1} , and S_{k-1} is renamed, after which it satisfies the following condition.

Condition 23. $S_{k-1} = S_{k-1} \cap H^{NW+} \cap H^{SE-}$.

Unless otherwise stated we assume from now on that Conditions 20, 21 and 23 are satisfied. It is also assumed that S_{k-1} is full (that is m -) dimensional. Recall also the standing assumption $d_1 = 1$.

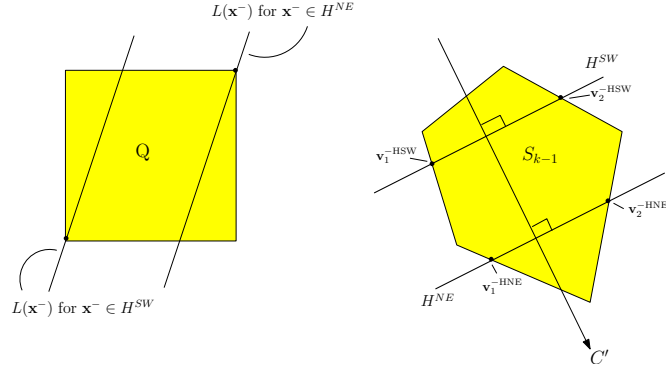


Figure 16: The hyperplanes H^{SW} and H^{NE} , and their associated primal lines.

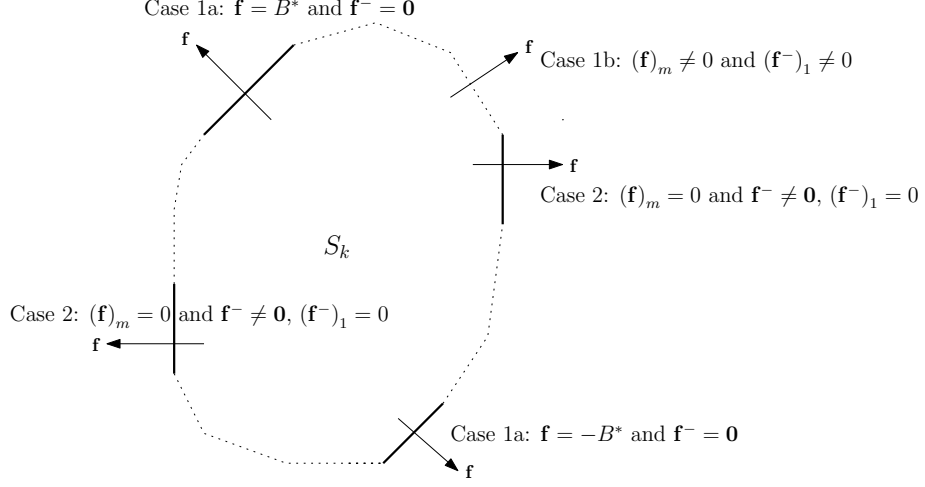


Figure 17: Classifying facets.

7.2 Facets

In this section we categorise the facets of S_k . The proofs of the following two theorems are in the Appendix.

Theorem 24. *Let F be a facet of S_k with direction \mathbf{f} , and choose any $\mathbf{x} \in \text{relint}(F)$. For any precursor \mathbf{x}^- of \mathbf{x} precisely one of the following conditions holds:*

1. $(\mathbf{f})_m \neq 0$, and either
 - (a) $\mathbf{f} = \pm B^*$ and $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-) \implies \mathbf{f}^- = \mathbf{0}$, with $\mathbf{x}^- \in \text{int}(S_{k-1}) \cap H^{NE+}$ if $\mathbf{f} = +B^*$, and $\mathbf{x}^- \in \text{int}(S_{k-1}) \cap H^{SW+}$ if $\mathbf{f} = -B^*$, or
 - (b) there is a facet F^- of S_{k-1} , with direction \mathbf{f}^- , for which $(\mathbf{f}^-)_1 \neq 0$ and $\mathbf{x}^- \in \text{relint}(F^-)$.
2. $(\mathbf{f})_m = 0$, and either
 - (a) there is a facet F^- of S_{k-1} , with direction \mathbf{f}^- , for which $(\mathbf{f}^-)_1 = 0$ and $\mathbf{x}^- \in \text{relint}(F^-)$, or
 - (b) there is a ridge R of S_{k-1} , where $\mathbf{x}^- \in \text{relint}(R)$, and there exists a direction $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ satisfying $(\mathbf{f}^-)_1 = 0$.

The cases are illustrated in Fig. 17. The splitting of cases rests on two observations:

1. For any \mathbf{f}^- there are three mutually exclusive possibilities: $\mathbf{f}^- = \mathbf{0}$, or $(\mathbf{f}^-)_1 \neq 0$, or $\mathbf{f}^- \neq \mathbf{0}$ and $(\mathbf{f}^-)_1 = 0$.
2. For any $\mathbf{f} \neq \mathbf{0}$ there are two mutually exclusive possibilities: $(\mathbf{f})_m = 0$ or $(\mathbf{f})_m \neq 0$.

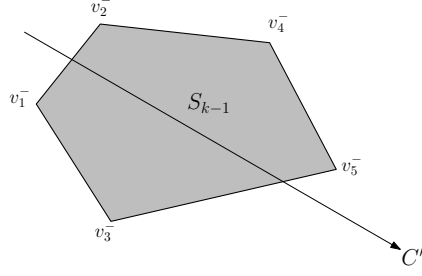


Figure 18: The ordering of vertices.

7.3 Vertices

Theorem 24 classifies the facets of S_k . The next result does the same for vertices.

Theorem 25. *Let \mathbf{v} be a vertex of S_k . Then one of the following cases holds:*

1. *Either $\mathbf{v} = g^T(\mathbf{v}^-)$ where $\mathbf{v}^- \in \mathbf{V}^{\text{HNE}}$, or $\mathbf{v} = g^B(\mathbf{v}^-)$ where $\mathbf{v}^- \in \mathbf{V}^{\text{HSW}}$, or*
2. *\mathbf{v} is equal to one of $g^T(\mathbf{v}^-)$, $g^B(\mathbf{v}^-)$, $g^L(\mathbf{v}^-)$ or $g^R(\mathbf{v}^-)$, where \mathbf{v}^- is a vertex of S_{k-1} .*

8 Geometric and combinatorial description of S_{k-1}

Three matrices, \mathbf{V} , \mathbf{F} and \mathcal{I} are used to describe S_{k-1} , and the algorithm updates all of them. Geometric data are contained in \mathbf{V} and \mathbf{F} , while combinatorial information on the structure of S_{k-1} is contained in \mathcal{I} . We use calligraphic font for incidence matrices, such as \mathcal{I} , whose entries are all zeros or ones.

1. The vertices in S_{k-1} are ordered so that, for $\text{vert}(S_{k-1}) = (\mathbf{v}_1^-, \dots, \mathbf{v}_{n_v}^-)$, there holds $C\mathbf{v}_{j_1}^- \geq C\mathbf{v}_{j_2}^-$ if $j_1 > j_2$. The ordering of ties is not important, but must be fixed. The m by n_v *vertex matrix* \mathbf{V} has columns containing the coordinates of the vertices of S_{k-1} , ordered in the same way. See Fig. 18.
2. The m by n_f *facet matrix* \mathbf{F} has columns containing the coordinates of the directions of the facets of S_{k-1} . Thus $\mathbf{F} = [\mathbf{f}_1^-, \dots, \mathbf{f}_{n_f}^-]$, where \mathbf{f}_i^- is a column vector representing the direction of facet number i . The magnitude of \mathbf{f}_i^- is not important, that is, for any $\alpha > 0$, $\alpha\mathbf{f}_i^-$ can be used in place of \mathbf{f}_i^- . The ordering of the facets is arbitrary but fixed, and the facet numbered i is denoted F_i .
3. The *vertex-facet incidence matrix* of S_{k-1} is the matrix $\mathcal{I} \in \{0, 1\}^{n_f \times n_v}$ which has entry $\mathcal{I}(i, j) = 1$ if $\mathbf{v}_j^- \in F_i$, and $\mathcal{I}(i, j) = 0$ otherwise. So F_i is the convex hull of the vertices identified by the ones in the i^{th} row of \mathcal{I} .

8.1 A simple example

In parallel with the development of the general procedure we give an example.

Example 26. *We illustrate recursive determination of the uncertainty set for the second order plant given by $\hat{n} = 2 - 7\lambda/6 + \lambda^2/3$ and $\hat{d} = 1 + 8\lambda/35 - 16\lambda^2/35$, so $m = 2$.*

By (4)

$$A = \begin{bmatrix} 0 & 1 \\ 16/35 & -8/35 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = [131/105 \quad -341/210], \quad \tilde{D} = 2.$$

The measurement $z_k = 2$, so $s^{\text{NW}} = -1$, $s^{\text{NE}} = 1$, $s^{\text{SW}} = 3$ and $s^{\text{SE}} = 5$.

The uncertainty set S_{k-1} , chosen to be simple yet capable of illustrating most modes of propagation, has three vertices and facets:

$$\mathbf{V} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix},$$

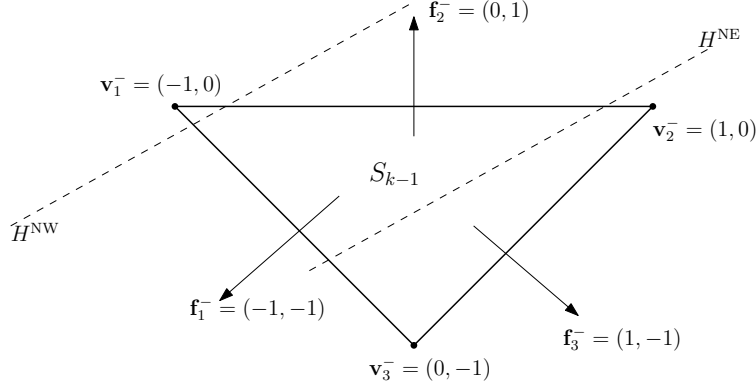


Figure 19: The uncertainty set at time $k - 1$ for Example 26

and the incidence matrix is

$$\mathcal{I} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

See Fig. 19, which depicts S_{k-1} and the hyperplanes (lines) H^{NW} and H^{NE} . ■

9 The incidence matrix for S_k

The rest of the paper is concerned with constructing \mathcal{I}_k , the incidence matrix for S_k , shown in Table 1. Initially \mathcal{I}_k is put equal to the $(2 + 5n_f + n_R)$ by $(4n_v + n_v^{NE} + n_v^{SW})$ zero matrix, that is \mathcal{I}_k is allocated sufficient space to accommodate, according to Theorems 24 and 25, all conceivable facets and vertices of S_k .

Each row of \mathcal{I}_k points to a potential facet of S_k , and each column points to a potential vertex of S_k . Case 1a) of Theorem 24 corresponds to the first two rows of Table 1, Case 1b) to rows three to six, Case 2a) to row seven, and Case 2b) to row eight. Thus the first two rows of Table 1 are reserved for facets propagated from the interior of S_{k-1} . The third (resp fourth, fifth, sixth) row of Table 1 is reserved for facets, with direction \mathbf{f} satisfying $(\mathbf{f})_m \neq 0$, which have been propagated up (resp down, left, right) from facets of S_{k-1} . The seventh row of Table 1 is reserved for facets of S_k which are propagations of facets of S_{k-1} whose directions \mathbf{f}^- satisfy $(\mathbf{f}^-)_1 = 0$, and the eighth row is reserved for facets F with the property that, for all $\mathbf{x} \in F$, precursors of \mathbf{x} belong to a ridge of S_{k-1} .

Turning now to the columns of Table 1, Case 1) of Theorem 25 corresponds to the first four columns, and Case 2) corresponds to columns five and six. Thus the first (resp second, third, fourth) column of Table 1 is reserved for vertices of S_k which are up (resp down, left, right) propagations of vertices of S_{k-1} . The fifth column is reserved for the vertices $g^B(\mathbf{V}^{HSW})$ of S_k , the sixth column for the vertices $g^T(\mathbf{V}^{HNE})$. If $H^{NE} \cap S_{k-1}$ is empty then $n_v^{NE} = 0$ and the column in Table 1 headed by $g^T(\mathbf{V}^{HNE})$ disappears, and a similar comment applies to the column headed by $g^B(\mathbf{V}^{HSW})$.

Some of the entries in \mathcal{I}_k are necessarily zero by virtue of Proposition 14, and these are indicated in Table 1 with boldface zeros. The other entries are comprised of twenty incidence matrices to be defined in Sections 14, 15 and 16. After \mathcal{I}_k has been constructed in accordance with Table 1 it must have some, perhaps most, rows and columns composed entirely of zeros. The final step is for these rows and columns of zeros to be deleted, and the extent to which \mathcal{I}_k shrinks in size depends on the value of z_k .

10 Intersection of S_{k-1} with hyperplanes

There are two geometric tasks we need to perform on S_{k-1} : intersection with special halfspaces and determination of special ridges. This section discusses the former, and the latter is covered in Section 16.

Facet directions \ Vertices	$g^T(\mathbf{V})$	$g^B(\mathbf{V})$	$g^L(\mathbf{V})$	$g^R(\mathbf{V})$	$g^B(\mathbf{V}^{\text{HSW}})$	$g^T(\mathbf{V}^{\text{HNE}})$
$+B^*$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
$-B^*$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$
$A^*\mathbf{f}_i^-(\text{Up})$	\mathcal{I}^T	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathcal{I}^{\text{HNET}}$
$A^*\mathbf{f}_i^-(\text{Down})$	$\mathbf{0}$	\mathcal{I}^B	$\mathbf{0}$	$\mathbf{0}$	$\mathcal{I}^{\text{HSWB}}$	$\mathbf{0}$
$\bar{A}^*\mathbf{f}_i^-(\text{Left})$	$\mathbf{0}$	$\mathbf{0}$	\mathcal{I}^L	$\mathbf{0}$	$\mathcal{I}^{\text{HSWL}}$	$\mathbf{0}$
$\bar{A}^*\mathbf{f}_i^-(\text{Right})$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	\mathcal{I}^R	$\mathbf{0}$	$\mathcal{I}^{\text{HNER}}$
$A^*\mathbf{f}_i^-(\text{Origin})$	$\mathcal{I}O^T$	$\mathcal{I}O^B$	$\mathcal{I}O^L$	$\mathcal{I}O^R$	$\mathcal{I}O^{\text{HSW}}$	$\mathcal{I}O^{\text{HNE}}$
$A^*\mathbf{f}_l^R(\text{Ridges})$	$\mathcal{I}R^T$	$\mathcal{I}R^B$	$\mathcal{I}R^L$	$\mathcal{I}R^R$	$\mathcal{I}R^{\text{HSW}}$	$\mathcal{I}R^{\text{HNE}}$

Table 1: The incidence matrix for S_k prior to the removal of rows and columns composed entirely of zeros

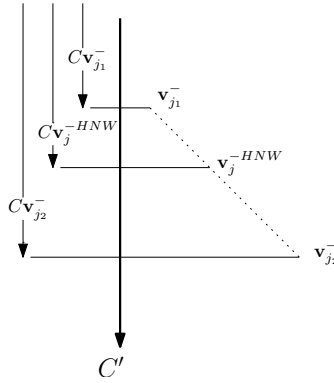


Figure 20: $\mathbf{v}_j^{-\text{HNW}}$ is a convex combination of $\mathbf{v}_{j_1}^-$ and $\mathbf{v}_{j_2}^-$, and $C\mathbf{v}_j^{-\text{HNW}} = s^{\text{NW}}$.

As explained in Section 7.1 the first step in the algorithm determines $S_{k-1} \cap H^{\text{NW}+} \cap H^{\text{SE}-}$, and we illustrate the intersection of a polytope with a halfspace by finding $S_{k-1} \cap H^{\text{NW}+}$. If H^{NW} cuts S_{k-1} we first determine the polytope $S_{k-1} \cap H^{\text{NW}}$, which is a facet of $S_{k-1} \cap H^{\text{NW}+}$ with direction $-C'$, a normal to H^{NW} .

Computing the intersection of polytopes with hyperplanes or half spaces is a well-known problem in computational geometry with an extensive literature. In the uncertainty set setting it has been treated in [20]. The best method to use depends very much on the available description of the polytope, and we are in the fortunate position of having at our disposal both geometric and combinatorial data.

The vertices of $S_{k-1} \cap H^{\text{NW}+}$ are the points of intersection of the edges of S_{k-1} with the hyperplane H^{NW} . To find the edges of S_{k-1} when $m < 5$ we use the fact that two vertices, $\mathbf{v}_{j_1}^-$ and $\mathbf{v}_{j_2}^-$, are connected by an edge if and only if there are at least $m - 1$ different facets that contain them both. When $m \geq 5$ it must additionally be checked that no other vertex is contained in all the facets that contain $\mathbf{v}_{j_1}^-$ and $\mathbf{v}_{j_2}^-$. In either case the edges of S_{k-1} can be efficiently found using \mathcal{I} alone, see Chapter 2 of [33] for a discussion of the relevant theory. An edge of S_{k-1} will intersect H^{NW} if the two vertices joining the edge, $\mathbf{v}_{j_1}^-$ and $\mathbf{v}_{j_2}^-$, satisfy $C\mathbf{v}_{j_1}^- < s^{\text{NW}} < C\mathbf{v}_{j_2}^-$. Thus all edges of S_{k-1} which intersect with H^{NW} can be efficiently found from \mathcal{I} , \mathbf{V} and s^{NW} . The points of intersection, $\mathbf{v}_j^{-\text{HNW}}$, are then given by linear interpolation:

$$\mathbf{v}_j^{-\text{HNW}} = \lambda \mathbf{v}_{j_1}^- + (1 - \lambda) \mathbf{v}_{j_2}^-$$

where

$$\lambda = (s^{\text{NW}} - C\mathbf{v}_{j_2}^-) / C(\mathbf{v}_{j_1}^- - \mathbf{v}_{j_2}^-).$$

See Fig. 20, which shows how $\mathbf{v}_j^{-\text{HNW}}$ is constructed from known $\mathbf{v}_{j_1}^-$, $\mathbf{v}_{j_2}^-$ and s^{NW} .

In Example 26, $C\mathbf{v}_1^- < s^{\text{NW}} < C\mathbf{v}_2^- < C\mathbf{v}_3^-$. Then $\mathcal{I}(1, 1) = \mathcal{I}(1, 3) = 1$ and $\mathcal{I}(2, 1) = \mathcal{I}(2, 2) = 1$ implies that there are precisely two edges of S_{k-1} which intersect H^{NW} , one joining the vertices \mathbf{v}_1^- and \mathbf{v}_2^- , and the other the vertices \mathbf{v}_1^- and \mathbf{v}_3^- . Linear interpolation gives $\mathbf{v}_1^{-\text{HNW}} = (-105/131, 0)$ and $\mathbf{v}_2^{-\text{HNW}} = (-551/603, -52/603)$ as the points of intersection of H^{NW} with two edges of S_{k-1} .

Vertex incidence vectors	Example 26, Fig. 19
$\mathcal{IV}^{\text{LL}}(j) = 1$ if $C\mathbf{v}_j^- < s^{\text{NW}}$	(1, 0, 0)
$\mathcal{IV}^{\text{NW}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{NW}}$	(0, 0, 0)
$\mathcal{IV}^{\text{T}}(j) = 1$ if $s^{\text{NW}} < C\mathbf{v}_j^- < s^{\text{NE}}$	(0, 0, 0)
$\mathcal{IV}^{\text{L}}(j) = 1$ if $s^{\text{NW}} < C\mathbf{v}_j^- < s^{\text{SE}}$	(0, 1, 1)
$\mathcal{IV}^{\text{NE}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{NE}}$	(0, 0, 0)
$\mathcal{IV}^{\text{SW}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{SW}}$	(0, 0, 0)
$\mathcal{IV}^{\text{B}}(j) = 1$ if $s^{\text{SW}} < C\mathbf{v}_j^- < s^{\text{SE}}$	(0, 0, 0)
$\mathcal{IV}^{\text{R}}(j) = 1$ if $s^{\text{NE}} < C\mathbf{v}_j^- < s^{\text{SE}}$	(0, 1, 1)
$\mathcal{IV}^{\text{SE}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{SE}}$	(0, 0, 0)
$\mathcal{IV}^{\text{RR}}(j) = 1$ if $C\mathbf{v}_j^- > s^{\text{SE}}$	(0, 0, 0)

Table 2: Vertex incidence vectors: How $L(\mathbf{v}_j^-)$ intersects Q , where $\mathbf{v}_j^- \in \text{vert}(S_{k-1})$ for $j = 1, \dots, n_v$

10.1 Incidence matrices for the intersection of hyperplanes with S_{k-1}

The incidence matrix \mathcal{I}^{HNW} identifies which vertices in \mathbf{V}^{HNW} are located in which facets of S_{k-1} .

More precisely $\mathcal{I}^{\text{HNW}} \in \{0, 1\}^{n_f \times n_v^{\text{NW}}}$ has entry $\mathcal{I}^{\text{HNW}}(i, j) = 1$ if the i^{th} facet of S_{k-1} contains the j^{th} vertex of \mathbf{V}^{HNW} , and $\mathcal{I}^{\text{HNW}}(i, j) = 0$ otherwise. The incidence matrices \mathcal{I}^{HSE} , \mathcal{I}^{HNE} and \mathcal{I}^{HSW} are defined similarly.

11 Notation classifying vertices and facets

The twenty incidence matrices of Table 1 are constructed from the vertex-facet incidence matrix of S_{k-1} and the incidence vectors for the vertices and facets of S_{k-1} , defined next.

11.1 Notation for the manner in which, for a vertex $\mathbf{v}_j^- \in S_{k-1}$, the line $L(\mathbf{v}_j^-)$ intersects the boundary of Q

The $1 \times n_v$ incidence vector \mathcal{IV}^{NW} points to vertices $\mathbf{v}_j^- \in S_{k-1}$ whose lines $L(\mathbf{v}_j^-)$ intersect the North West corner of Q . This occurs when $C\mathbf{v}_j^- = s^{\text{NW}}$. So $\mathcal{IV}^{\text{NW}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{NW}}$, with $\mathcal{IV}^{\text{NW}}(j) = 0$ otherwise. Likewise $\mathcal{IV}^{\text{NE}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{NE}}$, with $\mathcal{IV}^{\text{NE}}(j) = 0$ otherwise; $\mathcal{IV}^{\text{SW}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{SW}}$, with $\mathcal{IV}^{\text{SW}}(j) = 0$ otherwise; and $\mathcal{IV}^{\text{SE}}(j) = 1$ if $C\mathbf{v}_j^- = s^{\text{SE}}$, with $\mathcal{IV}^{\text{SE}}(j) = 0$ otherwise.

\mathcal{IV}^{T} points to those \mathbf{v}_j^- for which $L(\mathbf{v}_j^-)$ intersects the interior of the top side of Q . So $\mathcal{IV}^{\text{T}}(j) = 1$ if $s^{\text{NW}} < C\mathbf{v}_j^- < s^{\text{NE}}$, and $\mathcal{IV}^{\text{T}}(j) = 0$ otherwise. Likewise $\mathcal{IV}^{\text{B}}(j) = 1$ if $s^{\text{SW}} < C\mathbf{v}_j^- < s^{\text{SE}}$, with $\mathcal{IV}^{\text{B}}(j) = 0$ otherwise; $\mathcal{IV}^{\text{L}}(j) = 1$ if $s^{\text{NW}} < C\mathbf{v}_j^- < s^{\text{SW}}$, with $\mathcal{IV}^{\text{L}}(j) = 0$ otherwise; and $\mathcal{IV}^{\text{R}}(j) = 1$ if $s^{\text{NE}} < C\mathbf{v}_j^- < s^{\text{SE}}$, with $\mathcal{IV}^{\text{R}}(j) = 0$ otherwise.

We also define two incidence vectors indicating when $L(\mathbf{v}_j^-)$ does not intersect Q at all. Put $\mathcal{IV}^{\text{LL}}(j) = 1$ if $L(\mathbf{v}_j^-)$ lies to the left of Q . Thus $\mathcal{IV}^{\text{LL}}(j) = 1$ if $C\mathbf{v}_j^- < s^{\text{NW}}$, and $\mathcal{IV}^{\text{LL}}(j) = 0$ otherwise. Likewise put $\mathcal{IV}^{\text{LL}}(j) = 1$ if $C\mathbf{v}_j^- > s^{\text{SE}}$, and $\mathcal{IV}^{\text{LL}}(j) = 0$ otherwise. These definitions are summarised in Table 2, in which the vertex incidence vectors for Example 26 are shown in the final column.

11.2 For a facet having direction \mathbf{f}_i^- , notation for the intersections of the line $L^*(\mathbf{f}_i^-)$ with the coordinate axes y_k^* and u_k^*

In constructing the incidence matrix for S_k the only information needed concerning directions \mathbf{f}^- of facets of S_{k-1} is whether the dual lines $L^*(\mathbf{f}^-)$ intersect the positive or negative y_k^* and u_k^* axes. This is a consequence of the fact that it is only the signs of u_k^* and y_k^* , and not their magnitudes, that appear in the alignment conditions (19) and (20).

The $n_f \times 1$ vector \mathcal{IF}^{T} points to facets whose dual lines in the $y_k^* u_k^*$ plane intersect the positive u_k^* axis. Explicitly, $\mathcal{IF}^{\text{T}}(i) = 1$ if \mathbf{f}_i^- satisfies $u_k^*|_{y_k^*=0} > 0$ for $(y_k^*, u_k^*) \in L^*(\mathbf{f}_i^-)$, and $\mathcal{IF}^{\text{T}}(i) = 0$ otherwise. Likewise, the vectors \mathcal{IF}^{B} , \mathcal{IF}^{L} and \mathcal{IF}^{R} point to facets whose dual lines intersect respectively the negative u_k^* , negative y_k^*

Facet incidence vectors	Example 26, Fig. 21
$\mathcal{IF}^T(i) = 1$ if $u_k^* _{y_k^*=0} > 0$	(0,0,0,1)
$\mathcal{IF}^B(i) = 1$ if $u_k^* _{y_k^*=0} < 0$	(1,1,0,0)
$\mathcal{IF}^L(i) = 1$ if $y_k^* _{u_k^*=0} < 0$	(0,0,0,1)
$\mathcal{IF}^R(i) = 1$ if $y_k^* _{u_k^*=0} > 0$	(1,1,0,0)
$\mathcal{IF}^O(i) = 1$ if $u_k^* _{y_k^*=0} = 0$	(0,0,1,0)

Table 3: Facet incidence vectors: how $L^*(\mathbf{f}_i^-)$ intersects the coordinate axes in the $y_k^*u_k^*$ plane, where \mathbf{f}_i , $i = 1, \dots, n_f$, are the directions of the facets of S_{k-1}

and positive y_k^* axes, and \mathcal{IF}^O points to facets whose dual lines pass through the origin in the $y_k^*u_k^*$ plane. The definitions of the incidence vectors for the facets are given in the first column of Table 3.

Example 26 continued

The incidence vectors for the facets of S_{k-1} after intersection with $H^{\text{NW}+}$ are given in the second column of Table 3. ■

12 Removing states that are not propagated

As explained in Section 7.1 the first step in the algorithm is to remove from S_{k-1} all states that are not propagated. The construction of $S_{k-1} \cap H^{\text{NW}+} \cap H^{\text{SE}-}$ involves intersection with two halfspaces and we show details for one intersection, the other being similar. The vertex and facet matrices, \mathbf{V} and \mathbf{F} , and the incidence matrix \mathcal{I} , of $S_{k-1} \cap H^{\text{NW}+}$ are constructed as follows.

The vertices of S_{k-1} that are cut off by the hyperplane H^{NW} are deleted, and the vertices of the new facet are added:

$$\begin{aligned} \mathbf{V} \setminus \{\mathbf{v}_j^- : C\mathbf{v}_j^- < s^{\text{NW}}\} &\leftarrow \mathbf{V} \\ [\mathbf{V}^{\text{HSW}} \quad \mathbf{V}] &\leftarrow \mathbf{V} \end{aligned} \quad (23)$$

$$[-C' \quad \mathbf{F}] \leftarrow \mathbf{F} \quad (24)$$

The columns of \mathcal{I} pointing to vertices which have been deleted from \mathbf{F} are first deleted from \mathcal{I} . Then the new facet and its vertices are added:

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathcal{I}^{\text{HNW}} & \mathcal{I} \end{bmatrix} \leftarrow \mathcal{I} \quad (25)$$

The final step is to delete from \mathcal{I} all rows composed of zeros, because these rows denote facets of S_{k-1} which are not facets of $S_{k-1} \cap H^{\text{NW}+}$. These facets are also removed from \mathbf{F} , after which we have satisfaction of Condition 23, or equivalently $\mathcal{I}V^{\text{LL}} = \mathcal{I}V^{\text{RR}} = \mathbf{0}$.

Example 26 continued

We find a description of the polytope $S_{k-1} \cap H^{\text{NW}+} \cap H^{\text{SE}-}$ in Example 26.

Now

$$\mathbf{V}^{\text{HNW}} = \begin{bmatrix} -105/131 & -551/603 \\ 0 & -52/603 \end{bmatrix} \text{ and } \mathcal{I}^{\text{HNW}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

so the vertex, facet and incidence matrices for $S_{k-1} \cap H^{\text{NW}+}$ are:

$$\mathbf{V} = \begin{bmatrix} -105/131 & -551/603 & 1 & 0 \\ 0 & -52/603 & 0 & -1 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} -131/105 & -1 & 0 & 1 \\ 341/210 & -1 & 1 & -1 \end{bmatrix},$$

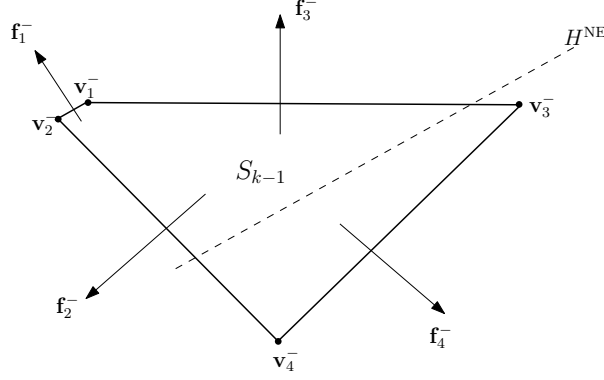


Figure 21: The uncertainty set S_{k-1} after intersection with H^{NW+}

Vertex incidence vectors	Example 26, Fig. 21
\mathcal{IV}^{NW}	$(1, 1, 0, 0)$
\mathcal{IV}^L	$(0, 0, 1, 1)$
\mathcal{IV}^R	$(0, 0, 1, 1)$

Table 4: Non-zero vertex incidence vectors for S_{k-1} in Example 26 after intersection with H^{NW+}

and

$$\mathcal{I} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The hyperplane H^{SE} does not intersect S_{k-1} , so $S_{k-1} \cap H^{NW+} \cap H^{SE-} = S_{k-1} \cap H^{NW+}$. The modified S_{k-1} is depicted in Fig. 21, and the non-zero incidence vectors for the modified S_{k-1} are given in Table 4. ■

13 Theorem 24, Case 1a): Interior to boundary propagation

The equation

$$\mathbf{f} = A^* \mathbf{f}^- + B^* y_k^* \quad (26)$$

occurring in Theorems 9 and 10 relates vectors in the normal cone of a point in S_{k-1} with directions in the normal cone of its successors. If F is a facet of S_k , $\mathbf{x} \in F$ and $\mathbf{f} = \pm B^* \in \mathcal{N}_{S_k}(\mathbf{x})$, so \mathbf{f} is the direction of F , there are two possibilities: $\mathbf{f}^- = \mathbf{0}$ and $y_k^* \neq 0$, or $\mathbf{f}^- \neq \mathbf{0}$, in which case $\mathbf{f}^- = \pm C'$. This Section treats the former case, where points in $\text{int}(S_{k-1})$ propagate to points on the boundary of S_k . The key observation is that $\mathbf{x}^- \in \text{int}(S_{k-1})$ if and only if the set $\mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ contains only the null vector.

Now $M(\mathbf{x}^-, \mathbf{f}^- = \mathbf{0}, z_k)$ is always non-empty, because all points of intersection of $L(\mathbf{x}^-)$ with Q are aligned with the origin in the $y_k^* u_k^*$ plane, so $(u_k, 0) \in M(\mathbf{x}^-, \mathbf{f}^- = \mathbf{0}, z_k)$ for all $u_k \in \widetilde{M}(\mathbf{x}^-, z_k)$. However, unless $L(\mathbf{x}^-)$ intersects a corner of Q , the only point in $L^*(\mathbf{f}^-)$ that is aligned with intersections of $L(\mathbf{x}^-)$ with Q is the origin in the $y_k^* u_k^*$ plane, in which case $\mathbf{f} = A^* \mathbf{f}^- + B^* y_k^* = \mathbf{0} + \mathbf{0} = \mathbf{0}$, implying that any successor to \mathbf{x}^- is in the interior of S_k . Propagation to the boundary of S_k requires $y_k^* \neq 0$, implying also $u_k^* \neq 0$. In this case, assuming Condition 20 is satisfied, there is the possibility of alignment between points on $L(\mathbf{x}^-)$ and $L^*(\mathbf{0})$, but only if $L(\mathbf{x}^-)$ intersects Q^{NE} or Q^{SW} , which is equivalent to the condition $\mathbf{x}^- \in H^{NE} \cap S_{k-1}$ or $\mathbf{x}^- \in H^{SW} \cap S_{k-1}$.

The previous discussion is summarised in the following two Propositions. Recall that satisfaction of Condition 20 is assumed.

Proposition 27. 1. For all $\mathbf{x}^- \in H^{NE} \cap S_{k-1}$, the state $\mathbf{x} := A\mathbf{x}^- + B \in \partial S_k$ is a successor to \mathbf{x}^- and $B^* \in \mathcal{N}_{S_k}(\mathbf{x})$.

2. For all $\mathbf{x}^- \in H^{\text{SW}} \cap S_{k-1}$, the state $\mathbf{x} := A\mathbf{x}^- - B \in \partial S_k$ is a successor to \mathbf{x}^- and $-B^* \in \mathcal{N}_{S_k}(\mathbf{x})$.

Proposition 28. Suppose that $\mathbf{x} \in \partial S_k$ is a successor to $\mathbf{x}^- \in \text{int}(S_{k-1})$. Then either

1. $\mathcal{N}_{S_k}(\mathbf{x}) = \{\alpha B^*, \alpha \geq 0\}$, $\mathbf{x}^- \in H^{\text{NE}} \cap \text{int}(S_{k-1})$ and $\mathbf{x} = A\mathbf{x}^- + B$; or
2. $\mathcal{N}_{S_k}(\mathbf{x}) = \{-\alpha B^*, \alpha \geq 0\}$, $\mathbf{x}^- \in H^{\text{SW}} \cap \text{int}(S_{k-1})$ and $\mathbf{x} = A\mathbf{x}^- - B$.

Proposition 27 shows that each point in $H^{\text{NE}} \cap S_{k-1}$ and $H^{\text{SW}} \cap S_{k-1}$ propagates to at least one point on the boundary of S_k , and Proposition 28 shows that any point in the interior of S_{k-1} which propagates to the boundary of S_k must belong to $(H^{\text{NE}} \cup H^{\text{SW}}) \cap S_{k-1}$.

Corollary 29. Suppose \mathbf{x} is a successor to $\mathbf{x}^- \in \text{int}(S_{k-1})$. Then $\mathbf{x} \notin \text{vert}(S_k)$.

Proof. By Proposition 28, $\mathbf{x}^- \in \text{int}(S_{k-1})$ cannot have a successor \mathbf{x} for which $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{N}_{S_k}(\mathbf{x})$ if $\mathbf{f}_1 = \alpha \mathbf{f}_2$ for $\alpha \neq 0$, implying \mathbf{x} cannot be a vertex of S_k . \square

Corollary 29 is illustrated in Fig. 5.

Combining Propositions 18, 27 and 28 yields a complete understanding of interior to boundary state propagation.

Proposition 30. 1. Suppose H^{NE} cuts S_{k-1} . Then $g^{\text{T}}(H^{\text{NE}} \cap S_{k-1}) =: F^{\text{NE}}$ is a facet of S_k with direction $+B^*$. Furthermore $\text{vert}(F^{\text{NE}}) = g^{\text{T}}(\mathbf{V}^{\text{HNE}})$.

2. Suppose H^{SW} cuts S_{k-1} . Then $g^{\text{B}}(H^{\text{SW}} \cap S_{k-1}) =: F^{\text{SW}}$ is a facet of S_k with direction $-B^*$. Furthermore $\text{vert}(F^{\text{SW}}) = g^{\text{B}}(\mathbf{V}^{\text{HSW}})$.

The facets of S_k having direction $+B^*$ or $-B^*$ are seen to be significant, since it is only these facets which can contain states propagated from the interior of S_{k-1} .

Example 26 continued

The hyperplane H^{NE} cuts S_{k-1} , but H^{SW} does not. It can be readily checked that, for S_{k-1} as in Fig. 21,

$$\mathbf{V}^{\text{HNE}} = \begin{bmatrix} 105/131 & -131/603 \\ 0 & -472/603 \end{bmatrix} \text{ and } \mathcal{I}^{\text{HNE}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have identified our first facet of S_k . It has direction B^* , and its vertices are $g^{\text{T}}(\mathbf{V}^{\text{HNE}})$. The incidence matrix \mathcal{I}^{HNE} will be needed later to identify those vertices in $g^{\text{T}}(\mathbf{V}^{\text{HNE}})$ which are vertices of facets of S_k that intersect the facet with direction B^* . \blacksquare

14 Theorem 24, Case 1b): Facet to facet propagation with $(\mathbf{f}^-)_1 \neq 0$

Sometimes either a facet of S_{k-1} , or the intersection of a facet of S_{k-1} with a half plane, propagates to a facet of S_k . The direction of the facet being propagated may or may not have its first component equal to zero, and in this Section we treat the latter case.

Recall the notation T for the top side of the square Q , excluding the corners, and similarly B , L and R for the other sides. Similar terminology to Definition 16 is used to describe propagations of a facet of S_{k-1} , or of the intersection of a facet of S_{k-1} with a halfspace, or of the intersection of S_{k-1} with a hyperplane. Suppose F^- is a facet of S_{k-1} and F is a facet of S_k . Then F^- is said to propagate up to F if $F = g^{\text{T}}(H^{\text{NE}-} \cap F^-)$, F^- is said to propagate right to F if $F = g^{\text{R}}(H^{\text{NE}+} \cap F^-)$, F^- is said to propagate down to F if $F = g^{\text{B}}(H^{\text{SW}+} \cap F^-)$, and F^- is said to propagate left to F if $F = g^{\text{L}}(H^{\text{SW}-} \cap F^-)$. Also $H^{\text{NE}} \cap S_{k-1}$ is said to propagate up to F if $F = g^{\text{T}}(H^{\text{NE}} \cap S_{k-1})$, and $H^{\text{SW}} \cap S_{k-1}$ is said to propagate down to F if $F = g^{\text{B}}(H^{\text{SW}} \cap S_{k-1})$. Here $g^{\text{T}}(S)$, for example, denotes the set $\{g^{\text{T}}(\mathbf{x}^-) : \mathbf{x}^- \in S\}$.

Suppose F^- is a facet of S_{k-1} with direction \mathbf{f}^- satisfying $(\mathbf{f}^-)_1 \neq 0$, so $L^*(\mathbf{f}^-)$ does not intersect the origin in the $u_k^* y_k^*$ plane. Suppose H^{NE} cuts F^- . Then, for all $\mathbf{x}^- \in F^- \cap H^{\text{NE}-}$, $L(\mathbf{x}^-)$ intersects $Q^{\text{NW}} \cup T \cup Q^{\text{NE}}$ in the $y_k u_k$ plane and, if $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is non-empty for $\mathbf{x}^- \in F^- \cap H^{\text{NE}-}$, that is if $L^*(\mathbf{f}^-)$ intersects the positive $u_k^* y_k^*$ axis then, by Theorem 9, \mathbf{x}^- propagates up to $g^{\text{T}}(\mathbf{x}^-)$ and $A^* \mathbf{f}^- \in \mathcal{N}_{S_k}(g^{\text{T}}(\mathbf{x}^-))$. Since $F^- \cap H^{\text{NE}-}$ is full dimensional (that is, has dimension $m-1$) then, by Theorem 9 and Proposition 18, $g^{\text{T}}(\mathbf{x}^-)$ belongs to a facet

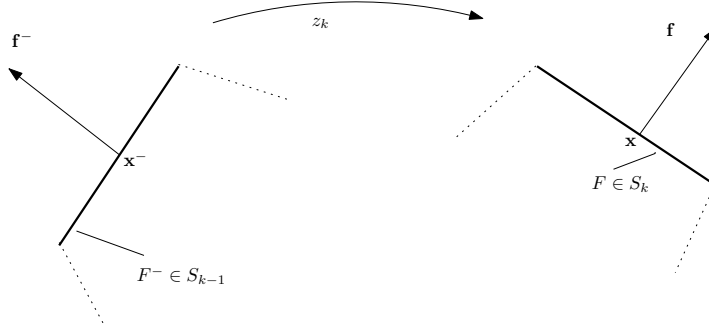


Figure 22: Illustration for Proposition 31, where $\mathbf{x} = g^T(\mathbf{x}^-)$ is a successor to \mathbf{x}^- , $\mathbf{f} = A^*\mathbf{f}^-$, and $(1, 0) \in M(\mathbf{x}^-, \mathbf{f}^-, z_k)$.

of S_k with direction $A^*\mathbf{f}^-$. Similar reasoning applies to the cases where $L(\mathbf{x}^-)$ intersects the bottom of Q , while for the left and right sides of Q (26) is replaced by $\mathbf{f} = \bar{A}^*\mathbf{f}^- + \bar{B}^*u_k^*$ (see (15), Section 2.3), and $u_k^* = 0$. This discussion is summarised in the following Proposition, where in Cases 3) and 4) \bar{A}^* is given by (16).

Proposition 31. Suppose F^- is a facet of S_{k-1} with direction \mathbf{f}^- satisfying $(\mathbf{f}^-)_1 \neq 0$.

1. If H^{NE} cuts F^- , and $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is non-empty for $\mathbf{x}^- \in F^- \cap H^{\text{NE}-}$, then $F^T := g^T(F^- \cap H^{\text{NE}-})$ is a facet of S_k with direction $A^*\mathbf{f}^-$. The vertices of F^T are $g^T(\text{vert}(F^- \cap H^{\text{NE}-}))$.
2. If H^{SW} cuts F^- , and $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is non-empty for $\mathbf{x}^- \in F^- \cap H^{\text{SW}+}$, then $F^B := g^B(F^- \cap H^{\text{SW}+})$ is a facet of S_k with direction $A^*\mathbf{f}^-$. The vertices of F^B are $g^B(\text{vert}(F^- \cap H^{\text{SW}+}))$.
3. If H^{SW} cuts F^- , and $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is non-empty for $\mathbf{x}^- \in F^- \cap H^{\text{SW}-}$, then $F^L := g^L(F^- \cap H^{\text{SW}-})$ is a facet of S_k with direction $\bar{A}^*\mathbf{f}^-$. The vertices of F^L are $g^L(\text{vert}(F^- \cap H^{\text{SW}-}))$.
4. If H^{NE} cuts F^- , and $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is non-empty for $\mathbf{x}^- \in F^- \cap H^{\text{NE}+}$, then $F^R := g^R(F^- \cap H^{\text{NE}+})$ is a facet of S_k with direction $\bar{A}^*\mathbf{f}^-$. The vertices of F^R are $g^R(\text{vert}(F^- \cap H^{\text{NE}+}))$.

Case 1), with $F^- \cap H^{\text{NE}-} = F^-$, is illustrated in Fig. 22.

Note that it is also possible that a facet of S_{k-1} with direction \mathbf{f}^- satisfying $(\mathbf{f}^-)_1 \neq 0$ may not propagate at all.

The incidence matrices which describe the four cases in Proposition 31 are given next.

14.1 The matrices \mathcal{I}^T , \mathcal{I}^B , \mathcal{I}^L and \mathcal{I}^R

The matrix $\mathcal{I}^T \in \{0, 1\}^{n_f \times n_v}$ is defined to have entry $\mathcal{I}^T(i, j) = 1$ if

1. $\mathcal{I}V^T(j) = 1$ or $\mathcal{I}V^{\text{NW}}(j) = 1$, that is $L(\mathbf{v}_j^-)$ intersects either the top side, or the North West corner, of Q , and
2. $\mathcal{I}(i, j) = 1$, that is the vertex $\mathbf{v}_j^- \in F_i$, implying $\mathbf{f}_i \in \mathcal{N}_{S_{k-1}}(\mathbf{v}_j^-)$ and
3. $\mathcal{I}F^T(i) = 1$, that is $L^*(\mathbf{f}_i)$ intersects the positive u_k^* axis, implying $M(\mathbf{v}_j^-, \mathbf{f}_i, z_k)$ is non-empty,

and $\mathcal{I}^T(i, j) = 0$ otherwise.

A one in the i^{th} row and j^{th} column of \mathcal{I}^T , for example, implies that there is a facet, F^T , of S_k with direction $A^*\mathbf{f}_i$, and that $g^T(\mathbf{v}_j^-) \in F^T$. This is a consequence of Proposition 31.

The matrices \mathcal{I}^B , \mathcal{I}^L and \mathcal{I}^R are defined similarly. The i^{th} row of \mathcal{I}^B points to the vertices of F_i which propagate down (with $u_k = -1$) to a facet of S_k having direction $A^*\mathbf{f}_i$. The i^{th} row of \mathcal{I}^L points to the vertices of F_i which propagate left (with $y_k = z_k - 1$) to a facet of S_k having direction $\bar{A}^*\mathbf{f}_i$. The i^{th} row of \mathcal{I}^R points to the vertices of F_i which propagate right (with $y_k = z_k + 1$) to a facet of S_k having direction $\bar{A}^*\mathbf{f}_i$.

Example 26 continued

The four by four matrices \mathcal{I}^T and \mathcal{I}^B are composed entirely of zeros; $\mathcal{I}^R(2, 4) = 1$, with $\mathcal{I}^R(i, j) = 0$ otherwise; and $\mathcal{I}^L(4, 3) = \mathcal{I}^L(4, 4) = 1$, with $\mathcal{I}^L(i, j) = 0$ otherwise. ■

The matrices $\mathcal{I}^{\text{HNET}}$, $\mathcal{I}^{\text{HNER}}$, $\mathcal{I}^{\text{HSWB}}$ and $\mathcal{I}^{\text{HSWL}}$ in Table 1 require incidence vectors, defined next, identifying facets and vectors of S_{k-1} that propagate up, down, left or right.

14.2 The incidence vectors $\mathcal{I}F^{\text{PT}}$, $\mathcal{I}F^{\text{PB}}$, $\mathcal{I}F^{\text{PL}}$ and $\mathcal{I}F^{\text{PR}}$

These vectors point to those facets of S_{k-1} that propagate up (resp. down, left, right).

The set $F_i \cap H^{\text{NE-}}$ is both full dimensional, and propagates up, if and only if there is at least one non-zero entry in the i^{th} row of \mathcal{I}^T . The $1 \times n_f$ vector $\mathcal{I}F^{\text{PT}}$ is defined by $\mathcal{I}F^{\text{PT}}(i) = 1$ if the i^{th} row of \mathcal{I}^T is not composed entirely of zeros, and $\mathcal{I}F^{\text{PT}}(i) = 0$ otherwise. Thus $\mathcal{I}F^{\text{PT}}$ points to those facets which propagate up, as desired.

The vectors $\mathcal{I}F^{\text{PB}}$, $\mathcal{I}F^{\text{PL}}$ and $\mathcal{I}F^{\text{PR}}$ are defined similarly.

Example 26 continued

$\mathcal{I}F^{\text{PT}} = \mathcal{I}F^{\text{PB}} = (0, 0, 0, 0)$, $\mathcal{I}F^{\text{PL}} = (0, 0, 0, 1)$ and $\mathcal{I}F^{\text{PR}} = (0, 1, 0, 0)$. ■

14.3 The incidence vectors $\mathcal{I}V^{\text{PT}}$, $\mathcal{I}V^{\text{PB}}$, $\mathcal{I}V^{\text{PL}}$ and $\mathcal{I}V^{\text{PR}}$

These vectors point to those vertices of S_{k-1} that both propagate up and lie on a facet propagated up. Thus $\mathcal{I}V^{\text{PT}}(j) = 1$ if the j^{th} column of \mathcal{I}^T is not composed entirely of zeros.

The vectors $\mathcal{I}V^{\text{PB}}$, $\mathcal{I}V^{\text{PL}}$ and $\mathcal{I}V^{\text{PR}}$ are defined similarly.

Example 26 continued

$\mathcal{I}V^{\text{PT}} = \mathcal{I}V^{\text{PB}} = (0, 0, 0, 0)$, $\mathcal{I}V^{\text{PL}} = (0, 0, 1, 1)$ and $\mathcal{I}V^{\text{PR}} = (0, 0, 0, 1)$. ■

14.4 The incidence matrices $\mathcal{I}^{\text{HNET}}$, $\mathcal{I}^{\text{HNER}}$, $\mathcal{I}^{\text{HSWB}}$ and $\mathcal{I}^{\text{HSWL}}$

It has been shown in Section 13 that the columns of $g^T(\mathbf{V}^{\text{HNE}})$ contain the coordinates of the vertices of any facet of S_k with direction B^* , and the columns of $g^B(\mathbf{V}^{\text{HSW}})$ contain the coordinates of the vertices of any facet of S_k with direction $-B^*$. We need to identify also the other facets of S_k which contain the vertices of $g^T(\mathbf{V}^{\text{HNE}})$ and $g^B(\mathbf{V}^{\text{HSW}})$.

The n_f by n_v^{HNE} incidence matrix $\mathcal{I}^{\text{HNET}}$ points to those vertices in \mathbf{V}^{HNE} that are in facets of S_{k-1} propagated up. Explicitly, $\mathcal{I}^{\text{HNET}}(i, j) = 1$ if $\mathbf{v}_j^- \in \mathbf{V}^{\text{HNE}} \cap F_i$ and F_i propagates up (so $\mathcal{I}F^{\text{PT}}(i) = 1$), and $\mathcal{I}^{\text{HNET}}(i, j) = 0$ otherwise. Thus the i^{th} row of $\mathcal{I}^{\text{HNET}}$ points to the vertices of $H^{\text{NE}} \cap F_i$ which propagate up. The matrices $\mathcal{I}^{\text{HNER}}$, $\mathcal{I}^{\text{HSWB}}$ and $\mathcal{I}^{\text{HSWL}}$ are defined similarly. For example $\mathcal{I}^{\text{HSWL}}$ is the n_f by n_v^{HSW} matrix defined by $\mathcal{I}^{\text{HSWL}}(i, j) = 1$ if $\mathbf{v}_j^- \in \mathbf{V}^{\text{HSW}} \cap F_i$ and F_i propagates left (so $\mathcal{I}F^{\text{PL}}(i) = 1$), $\mathcal{I}^{\text{HSWL}}(i, j) = 0$ otherwise.

Example 26 continued

$\mathcal{I}^{\text{HNET}}$ is the four by two matrix composed entirely of zeros, $\mathcal{I}^{\text{HNER}}$ is also a four by two matrix composed of zeros, except for the element $\mathcal{I}^{\text{HNER}}(2, 2) = 1$. The matrices $\mathcal{I}^{\text{HSWB}}$ and $\mathcal{I}^{\text{HSWL}}$ are empty because H^{SW} does not intersect S_{k-1} . ■

We have completed the first six rows of Table 1. The next section is concerned with the seventh row.

15 Theorem 24, Case 2a): Facet to facet propagation with $(\mathbf{f}^-)_1 = 0$

A facet of S_{k-1} whose direction has first component equal to zero always propagates to a facet of S_k . Suppose F^- is a facet of S_{k-1} with direction \mathbf{f}^- satisfying $(\mathbf{f}^-)_1 = 0$, so the line $L^*(\mathbf{f}^-)$ passes through the origin in the $y_k^* u_k^*$ plane. Not only do all points in F^- propagate to points in ∂S_k , each point $\mathbf{x}^- \in F^-$ for which $L(\mathbf{x}^-)$ intersects two sides of Q will propagate to an infinite number of points in S_k because the origin in the $y_k^* u_k^*$ plane is aligned with all points in Q . By Theorem 9 all of these propagated points belong to a facet of S_k with direction $A^* \mathbf{f}^-$.

Proposition 32. *Suppose F^- is a facet of S_{k-1} with direction \mathbf{f}^- satisfying $(\mathbf{f}^-)_1 = 0$. There is a facet of S_k , denoted F , with direction $\mathbf{f} := A^* \mathbf{f}^-$ where $(\mathbf{f})_m = 0$, and the vertices of F are $g^T(\text{vert}(F^- \cap H^{\text{NE-}}))$.*

The incidence vectors defined in the next two sections point to the precursors of vertices of facets whose direction vectors satisfy $(\mathbf{f})_m = 0$.

15.1 The incidence vectors \mathcal{IO}^T , \mathcal{IO}^B , \mathcal{IO}^L , and \mathcal{IO}^R

The matrix $\mathcal{IO}^T \in \{0, 1\}^{n_f \times n_v}$ has entry $\mathcal{IO}^T(i, j) = 1$ if $\mathbf{v}_j \in F_i$, $(\mathbf{f}_i)_1 = 0$, and $\mathcal{IV}^{\text{PT}}(j) = 1$. Otherwise $\mathcal{IO}^T(i, j) = 0$. Likewise, $\mathcal{IO}^L \in \{0, 1\}^{n_f \times n_v}$ has entry $\mathcal{IO}^L(i, j) = 1$ if $\mathbf{v}_j \in F_i$, $(\mathbf{f}_i)_1 = 0$, and $\mathcal{IV}^{\text{PL}}(j) = 1$. Otherwise $\mathcal{IO}^L(i, j) = 0$. The matrices \mathcal{IO}^B and \mathcal{IO}^R are defined similarly.

Example 26 continued

The matrices \mathcal{IO}^T , \mathcal{IO}^B and \mathcal{IO}^R are all composed entirely of zeros. The matrix \mathcal{IO}^L is composed of all zeros except for one component, namely $\mathcal{IO}^L(3, 3) = 1$. ■

15.2 The incidence matrices $\mathcal{IO}^{\text{HNE}}$ and $\mathcal{IO}^{\text{HSW}}$

It is possible that H^{NE} cuts a facet F^- of S_{k-1} with direction \mathbf{f}^- satisfying $(\mathbf{f}^-)_1 = 0$. The matrix $\mathcal{IO}^{\text{HNE}}$ identifies the vertices of $H^{\text{NE}} \cap S_{k-1}$ which belong to a facet with direction vector having first component equal to zero. Explicitly the matrix $\mathcal{IO}^{\text{HNE}} \in \{0, 1\}^{n_f \times n_v^{\text{HNE}}}$ has entry $\mathcal{IO}^{\text{HNE}}(i, j) = 1$ if $\mathcal{IF}^{\text{O}}(i) = 1$ and $\mathcal{I}^{\text{HNE}}(i, j) = 1$, and $\mathcal{IO}^{\text{HNE}}(i, j) = 0$ otherwise. Then, for example, $\mathcal{IO}^{\text{HNE}}$ points to those vertices of the polytope $H^{\text{NE}} \cap S_{k-1}$ that belong to a facet of S_{k-1} with direction vector whose first component is zero.

Likewise $\mathcal{IO}^{\text{HSW}} \in \{0, 1\}^{n_f \times n_v^{\text{HSW}}}$ has entry $\mathcal{IO}^{\text{HSW}}(i, j) = 1$ if $\mathcal{I}^{\text{O}}(i) = 1$ and $\mathcal{I}^{\text{HSW}}(i, j) = 1$, and $\mathcal{IO}^{\text{HSW}}(i, j) = 0$ otherwise.

Example 26 continued

$\mathcal{IO}^{\text{HNE}}(3, 1) = 1$ and all the other elements are zero. The matrix $\mathcal{IO}^{\text{HSW}}$ is empty. ■

16 Theorem 24, Case 2b): Propagation from a ridge in S_{k-1} to a facet in S_k

In Case 2b) of Theorem 24 facets of S_k are propagations of ridges of S_{k-1} . See Fig. 23 where S_{k-1} possesses two intersecting facets, with directions \mathbf{f}_1 and \mathbf{f}_2 , whose first components have opposite signs. These facets have the state \mathbf{x} in their intersection, so necessarily there exists a direction in the normal cone of \mathbf{x} whose first component is zero. We define R to be the set of all such ridges of S_{k-1} , and denote its cardinality by n_R . Clearly R is non-empty if S_{k-1} is non-empty and full-dimensional.

Definition 33. $R = \{R_l, l = 1, \dots, n_R : R_l \text{ is a ridge of } S_{k-1} \text{ formed from the intersection of two facets, } F_1 \text{ and } F_2 \text{ say, for which } (\mathbf{f}_1)_1 < 0 \text{ and } (\mathbf{f}_2)_1 > 0, \text{ where } \mathbf{f}_1 \text{ and } \mathbf{f}_2 \text{ are the directions of } F_1 \text{ and } F_2\}$.

Associated with any ridge in R there is by definition a direction in the cone generated by \mathbf{f}_1 and \mathbf{f}_2 having the property that its first component is zero. This direction is clearly unique.

Definition 34. For any $R_l \in R$ define \mathbf{f}_l^R to be the direction in the convex hull of \mathbf{f}_1 and \mathbf{f}_2 for which $(\mathbf{f}_l^R)_1 = 0$.

It is readily shown that $\mathbf{f}_l^R \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ for all $\mathbf{x}^- \in R_l$.

Example 35. Fig. 24 illustrates ridge to facet propagation for the case where S_{k-1} is three dimensional, that is $m = 3$, and the one-dimensional ridge $R_l \in R$ has vertices \mathbf{v}_1^- , and \mathbf{v}_2^- . For all $\mathbf{x}^- \in R_l$ the line $L(\mathbf{x}^-)$ is parallel to and lies between the lines $L(\mathbf{v}_1^-)$ and $L(\mathbf{v}_2^-)$, and $M(\mathbf{x}^-, \mathbf{f}_l^R, z_k)$ is non-empty because the line $L^*(\mathbf{f}_l^R)$ passes through the origin in the $y_k^* u_k^*$ plane. Let $P(y_k, u_k)$ be any point in the shaded region of Q in Fig. 24. There exists $\mathbf{x}^- \in R_l$ for which $L(\mathbf{x}^-)$ passes through $P(y_k, u_k)$, and $(u_k, 0) \in M(\mathbf{x}^-, \mathbf{f}_l^R, z_k)$. It follows from Theorem 9 that $A\mathbf{x}^- + Bu_k \in \partial S_k$ and $A^*\mathbf{f}_l^R \in \mathcal{N}_{S_k}(A\mathbf{x}^- + Bu_k)$, for all $\mathbf{x}^- \in R_l$ and all $u_k \in \widetilde{M}(\mathbf{x}^-, z_k)$. We have shown that $\{A\mathbf{x}^- + Bu_k : \mathbf{x}^- \in R_l \text{ and } u_k \in \widetilde{M}(\mathbf{x}^-, z_k)\}$ is a facet of S_k , as indicated in Fig. 24, and that the direction of this facet is $A^*\mathbf{f}_l^R$.

The discussion in Example 35 applies more generally and in any dimension.

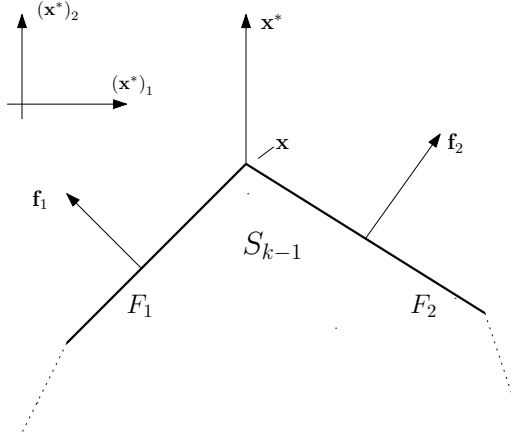


Figure 23: Two facets intersect in a ridge containing \mathbf{x} . The direction $\mathbf{x}^* \in \mathcal{N}_{S_{k-1}}(\mathbf{x})$ satisfies $(\mathbf{x}^*)_1 = 0$.

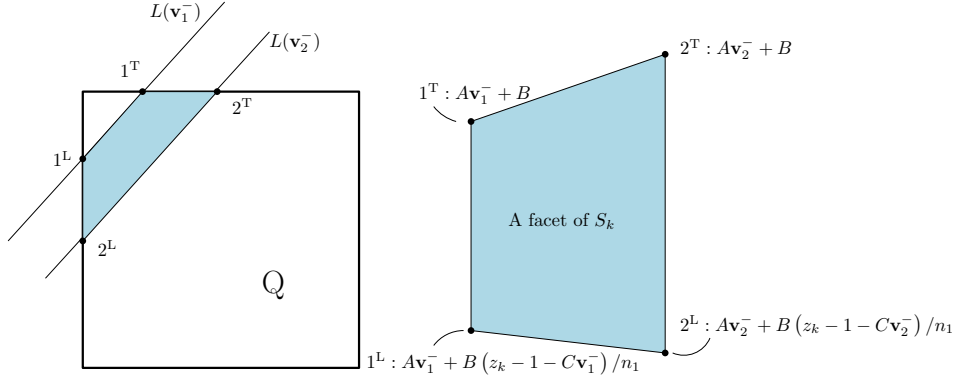


Figure 24: The points $\mathbf{v}_1^-, \mathbf{v}_2^- \in S_{k-1}$ are vertices of a ridge, $R_l \in R$, that is propagated to a facet of S_k . Every point in the facet can be identified with a point in the shaded part of Q .

Proposition 36. *For all $R_l \in R$ there is a facet of S_k with direction $A^* \mathbf{f}_l^R$.*

For any S_{k-1} and any $R_l \in R$, the direction \mathbf{f}_l^R can be found using linear interpolation. More precisely $\mathbf{f}_l^R = t\mathbf{f}_1 + (1-t)\mathbf{f}_2$ where $t = (\mathbf{f}_1)_1 / ((\mathbf{f}_2)_1 - (\mathbf{f}_1)_1)$, and \mathbf{f}_1 and \mathbf{f}_2 are the directions of the facets which intersect to form R_l . The facet of S_k with direction $A^* \mathbf{f}_l^R$ can be interpreted as a prismatic extension between two affine maps of R_l .

Example 26 continued

In Fig. 19, $n_R = 1$, $R_1 = \mathbf{v}_3^-$ and $\mathbf{f}_1^R = [0, -1]^T$. The facet of S_k which is a propagation of the ridge (vertex) \mathbf{v}_3^- has direction $A^* \mathbf{f}_1^R = [-1, 0]^T$. ■

The incidence matrices in the bottom row of Table 1 require some preliminary definitions, given next.

16.1 The incidence vectors \mathcal{IRV} , \mathcal{IR}^T , \mathcal{IR}^B , \mathcal{IR}^L and \mathcal{IR}^R

The vertex set of R_l is the intersection of the vertex sets of the facets which intersect to form R_l . The matrix $\mathcal{IRV} \in \{0, 1\}^{n_R \times n_v}$ has entry $\mathcal{IRV}(l, j) = 1$ if the ridge R_l contains \mathbf{v}_j , and $\mathcal{IRV}(l, j) = 0$ otherwise. So the l^{th} row of \mathcal{IRV} points to the vertex set of R_l .

The matrix $\mathcal{IR}^T \in \{0, 1\}^{n_R \times n_v}$ has entry $\mathcal{IR}^T(l, j) = 1$ if $\mathcal{IV}^{\text{PT}}(j) = 1$ and $\mathcal{IRV}(l, j) = 1$, and $\mathcal{IR}^T(l, j) = 0$ otherwise. So the l^{th} row of \mathcal{IR}^T points to vertices \mathbf{v}_j of R_l which propagate up. The matrices \mathcal{IR}^B , \mathcal{IR}^L and \mathcal{IR}^R are defined similarly.

It follows from Propositions 17 and 18 that a vertex of a facet of S_k propagated from a ridge of S_{k-1} must be either one of $g^T(\mathbf{x}^-)$, $g^B(\mathbf{x}^-)$, $g^L(\mathbf{x}^-)$ or $g^R(\mathbf{x}^-)$ where \mathbf{x}^- is a vertex of S_{k-1} , or $g^T(\mathbf{v}_j^{\text{HNE}})$ for some $\mathbf{v}_j^{\text{HNE}} \in \mathbf{V}^{\text{HNE}}$, or $g^B(\mathbf{v}_j^{\text{HSW}})$ for some $\mathbf{v}_j^{\text{HSW}} \in \mathbf{V}^{\text{HSW}}$. However, not all such points are guaranteed to be vertices of a facet propagated from a ridge of S_k . If a ridge intersects neither H^{NE} nor H^{SW} , then its vertices have precursors contained in $\mathcal{IR}^T \cup \mathcal{IR}^B \cup \mathcal{IR}^L \cup \mathcal{IR}^R$.

Example 26 continued

$\mathcal{IRV} = [0, 0, 0, 1]$, $\mathcal{IR}^T = \mathcal{IR}^B = [0, 0, 0, 0]$ and $\mathcal{IR}^L = \mathcal{IR}^R = [0, 0, 0, 1]$. Thus \mathbf{v}_4 propagates both left and right to vertices of the facet of S_k propagated from \mathbf{v}_4 , and these vertices are $g^L(\mathbf{v}_4)$ and $g^R(\mathbf{v}_4)$. ■

16.2 The incidence vectors \mathcal{IRF} , $\mathcal{IR}^{\text{HNE}}$ and $\mathcal{IR}^{\text{HSW}}$

Finally we come to the bottom right hand side of Table 1. This deals with propagation from points \mathbf{x}^- which both lie in a ridge of S_{k-1} and belong to one or both of the hyperplanes H^{NE} and H^{SW} , so $L(\mathbf{x}^-)$ passes through Q^{NE} or Q^{SW} and $\mathbf{0} \neq \mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ satisfies $(\mathbf{f}^-)_1 = 0$.

The matrix $\mathcal{IRF} \in \{0, 1\}^{n_R \times n_f}$ has entry $\mathcal{IRF}(l, i) = 1$ if the ridge R_l is contained in the i^{th} facet of S_{k-1} , and $\mathcal{IRF}(l, i) = 0$ otherwise. So the row $\mathcal{IRF}(l, :)$ points to the two facets whose intersection is the ridge R_l .

The matrix $\mathcal{IR}^{\text{HNE}} \in \{0, 1\}^{n_R \times n_v^{\text{NE}}}$ has entry $\mathcal{IR}^{\text{HNE}}(l, j) = 1$ if $\mathbf{v}_j \in \mathbf{V}^{\text{HNE}} \cap R_l$. Computationally, $\mathcal{IR}^{\text{HNE}}(l, j) = 1$ if, for any integers i_1 and i_2 satisfying $\mathcal{IRF}(i_1, l) = \mathcal{IRF}(i_2, l) = 1$, there holds $\mathcal{I}^{\text{HNE}}(i_1, j) = \mathcal{I}^{\text{HNE}}(i_2, j) = 1$, and $\mathcal{IR}^{\text{HNE}}(l, j) = 0$ otherwise. For $j = 1, \dots, n_v^{\text{NE}}$, $g^T(\mathbf{v}_j^{\text{HNE}})$ belongs to the facet of S_k propagated from the ridge R_l if the two facets in S_{k-1} intersecting to form R_l are included in the set of facets in S_{k-1} which intersect to form the edge containing $\mathbf{v}_j^{\text{HNE}}$.

The construction above relies on the fact that any vertex of the polytope $H^{\text{NE}} \cap R_l$, where R_l is a ridge of S_{k-1} , is a vertex of the polytope $H^{\text{NE}} \cap S_{k-1}$. So, if the two facets which $\mathcal{IRF}(l, :)$ points to are included in those facets that $\mathcal{I}^{\text{NE}}(:, j)$ points to, a one is placed in Table 1 at the intersection of the row pointing to the ridge R_l and the column pointing to \mathbf{v}_j^{NE} . Similar comments apply for the $n_R \times n_v^{\text{SW}}$ matrix $\mathcal{IR}^{\text{HSW}}$.

Example 26 continued

The ridge (vertex) \mathbf{v}_4 belongs to F_2 and F_4 , so $\mathcal{IRF} = [0, 1, 0, 1]$. The ridge intersects neither H^{SW} nor H^{NE} , so $\mathcal{IR}^{\text{HNE}} = [0, 0]$ and $\mathcal{IR}^{\text{HSW}}$ is empty. ■

17 Pseudocode

The algorithm takes the facets, vertices and vertex-facet incidence matrix of S_{k-1} , and the measurement z_k , as inputs; it outputs the facets, vertices and vertex-facet incidence matrix of S_k .

Algorithm Updating the uncertainty set

Input: The measurement z_k , the vertex matrix, \mathbf{V} , the facet matrix, \mathbf{F} , and the vertex-facet incidence matrix, \mathcal{I} , of the polytope S_{k-1} .

Output: The vertex matrix, \mathbf{V}_k , the facet matrix, \mathbf{F}_k , and the vertex-facet incidence matrix, \mathcal{I}_k , of the polytope S_k .

- 1: Sort the vertices in \mathbf{V} and classify the facets in \mathbf{F} , as explained in Section 8
- 2: $S_{k-1} \leftarrow S_{k-1} \cap H^{\text{NW}+} \cap H^{\text{SE}-}$, see Section 12
- 3: Initialise \mathcal{I}_k equal to the $(2 + 5n_f + n_R)$ by $(4n_v + n_v^{\text{HNE}} + n_v^{\text{HSW}})$ zero matrix
- 4: Populate the body of Table 1 with the twenty incidence matrices defined in Sections 14, 15 and 16
- 5: $\mathcal{I}_k \leftarrow \mathcal{I}_k$ with rows and columns composed entirely of zeros removed
- 6: Construct each column of \mathbf{F}_k by calculating, for each row of \mathcal{I}_k , the direction of the propagated facet
- 7: Construct each column of \mathbf{V}_k by calculating, for each column of \mathcal{I}_k , the vertex of S_k propagated from ∂S_{k-1}

Example 26 continued

The incidence matrix for S_k , constructed according to Table 1, is a sparse 23×18 matrix, with all except ten elements equal to zero. After the rows and columns composed entirely of zeros have been removed, S_k becomes a 5×5 matrix. The polytope S_k has five vertices and facets:

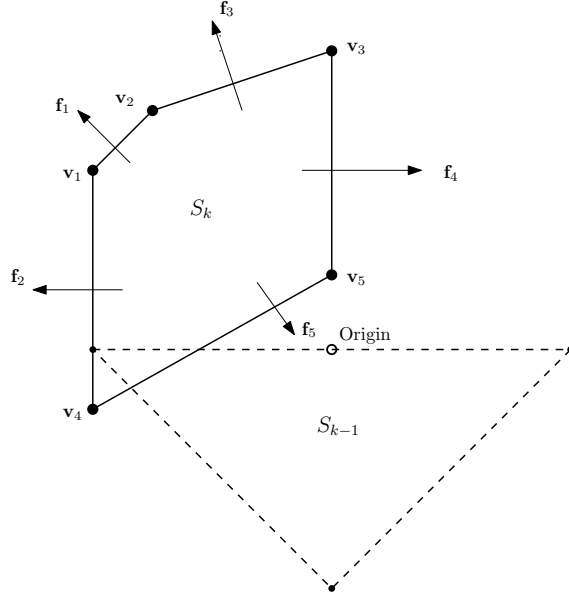


Figure 25: The polytopes S_{k-1} and S_k for Example 26.

$$\mathbf{V}_k = \begin{bmatrix} -1 & -472/603 & 0 & -1 & 0 \\ 11/12 & 217/201 & 179/131 & -1/12 & 1/3 \end{bmatrix}, \mathbf{F}_k = \begin{bmatrix} -9/2 & -1 & -1 & 1 & 5/2 \\ 6 & 0 & 131/48 & 0 & -6 \end{bmatrix},$$

and

$$\mathcal{I}_k = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

See Fig. 25 for a depiction of S_{k-1} and S_k . ■

18 CONCLUSIONS

The algorithm described in this paper, because most calculations use Boolean matrices and are therefore fast, has potential application to any problem requiring set-valued observers. In real-time applications existing exact techniques have limited applicability because at each time instant several linear programs must be solved to remove redundant constraints. Model falsification using set-valued observers, useful for example in fault detection and isolation (see [27] for example), is one area of current research that could perhaps make use of the algorithmic approach presented here. The classical problem of control under uncertainty is another.

One interesting question that can be investigated with the tools provided in this paper is how the complexity of the uncertainty set changes with time. Many authors have alluded to a potential problem here, yet there has been little published research on this issue. One of the very few publications to investigate this, on a related problem, is [20] which reports little or no increase in complexity after about fifty time steps. It is clear that, for any plant, the complexity of the uncertainty set after many measurements have been processed depends on the measurement sequence and the *a priori* information on the size of the bounds on the process and measurement noises. A vertex is destroyed if it has no successor that is a vertex, and has been created if it has no precursor that is a vertex. Vertices and facets can be both created and destroyed as new measurements arrive, and variability in the size of the measurements increases the rate of destruction. What happens for realistic measurement sequences and *a priori* bounds is a topic of research to which the insights provided by this paper can be applied. Intelligent selection of

any approximation scheme, if needed, and the timing of its possible incorporation into any application, will require such information.

More work is needed to extend the results to time-varying linear plants, and to multivariable systems. There does not seem to be any reason to believe this will not be possible. For time-varying plants the primal lines will no longer be parallel as time evolves, and the same statement holds true for dual lines, but checking alignment is not any more difficult. For multivariable plants alignment can be defined between vector inputs and outputs, although the computations will be more involved. The use of state and normal cone propagation in multivariable systems is a topic for future research.

19 Appendix

Proof of Theorem 12 After expressing the program $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ as an equivalent linear program, the standard duality result in asymmetric form ([17] p. 86, 96) is used:

$$\begin{array}{ll} \text{Dual} & \text{Primal} \\ \min \mathbf{c}^T \mathbf{x} & \max \boldsymbol{\lambda}^T \mathbf{b} \\ \text{s. t. } A\mathbf{x} = \mathbf{b} & \text{s. t. } A^T \boldsymbol{\lambda} \leq \mathbf{c} \\ \mathbf{x} \geq 0 & \end{array} \quad (27)$$

where *complementary slackness* holds: Let \mathbf{x} and $\boldsymbol{\lambda}$ be feasible solutions for the primal and dual problems, respectively. A necessary and sufficient condition that they both be optimal solutions is that for all i

- i) $x_i > 0 \implies a_i^T \boldsymbol{\lambda} = c_i$ (where a_i^T is the i^{th} row of A^T)
- ii) $x_i = 0 \Leftarrow a_i^T \boldsymbol{\lambda} < c_i$.

Note that we have swapped the labels for primal and dual from that given in [17], because it is the estimation program that we regard as the primal problem, and the estimation program is naturally expressed in the form of the maximization in (27). A related notational point is that the use of the symbol \mathbf{x} for the dual decision variable in (27) is different from the use of the symbols \mathbf{x}_0 , \mathbf{x}_0^* and \mathbf{x}_k , which retain their meanings given in the body of the paper.

The program $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ has a convex piecewise linear cost function and linear constraints. There is a standard procedure, which we now follow, for converting such a program to an equivalent linear programming problem. Introduce new non-negative k -dimensional column vectors \mathbf{u}^{*+} , \mathbf{u}^{*-} , \mathbf{y}^{*+} and \mathbf{y}^{*-} , and put $u_j^* = u_j^{*+} - u_j^{*-}$ and $y_j^* = y_j^{*+} - y_j^{*-}$. At optimality at least one of u_j^{*+} , u_j^{*-} , and at least one of y_j^{*+} , y_j^{*-} , will be zero, so $|u_j^*| = u_j^{*+} + u_j^{*-}$ and $|y_j^*| = y_j^{*+} + y_j^{*-}$. Since $\langle \mathbf{x}_0^*, \mathbf{x}_0 \rangle = -\mathbf{x}_0^T [N_U^T y_{1:m}^* + D_U^T u_{1:m}^*]$, the cost function for $\mathcal{D}_{z_{1:k}}(\mathbf{f})$, namely $\|\mathbf{y}^*\|_1 + \|\mathbf{v}^*\|_1 + \langle y_{1:k}^*, z_{1:k} \rangle + \langle \mathbf{x}_0^*, \mathbf{x}_0 \rangle =: J_{\text{dual}}$, can be written as

$$J_{\text{dual}} = [\mathbf{1}_{4k} + \boldsymbol{\delta} + \boldsymbol{\gamma}] \begin{bmatrix} \mathbf{y}^{*+} \\ \mathbf{y}^{*-} \\ \mathbf{u}^{*+} \\ \mathbf{u}^{*-} \end{bmatrix}$$

where $\mathbf{1}_{4k}$ denotes a $4k$ -dimensional row vector of ones, and the row vectors $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ are defined by

$$\begin{aligned} \boldsymbol{\delta} &:= \begin{bmatrix} -\mathbf{x}_0^T N_U^T & \mathbf{0}_{k-m} & \mathbf{x}_0^T N_U^T & \mathbf{0}_{k-m} & -\mathbf{x}_0^T D_U^T & \mathbf{0}_{k-m} & \mathbf{x}_0^T D_U^T & \mathbf{0}_{k-m} \end{bmatrix} \\ \boldsymbol{\gamma} &:= \begin{bmatrix} z_{1:k}^T & -z_{1:k}^T & \mathbf{0}_{2k} \end{bmatrix}, \end{aligned}$$

where $\mathbf{0}_{k-m}$ denotes a $(k-m)$ -dimensional row vector of zeros.

The constraints for the program $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ in terms of the new variables are

$$\begin{bmatrix} N_{k \times k}^T & -N_{k \times k}^T & D_{k \times k}^T & -D_{k \times k}^T \end{bmatrix} \begin{bmatrix} \mathbf{y}^{*+} \\ \mathbf{y}^{*-} \\ \mathbf{u}^{*+} \\ \mathbf{u}^{*-} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{f} \end{bmatrix}$$

$$y_j^{*+}, y_j^{*-}, u_j^{*+}, u_j^{*-} \geq 0.$$

In (27) put

$$\begin{aligned} A &= \begin{bmatrix} N_{k \times k}^T & -N_{k \times k}^T & D_{k \times k}^T & -D_{k \times k}^T \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} \mathbf{y}^{*+T} & \mathbf{y}^{*-T} & \mathbf{u}^{*+T} & \mathbf{u}^{*-T} \end{bmatrix}^T, \quad \mathbf{c}^T = \mathbf{1}_{4k} + \boldsymbol{\delta} + \boldsymbol{\gamma} \\ \mathbf{b} &= \begin{bmatrix} 0 & \dots & 0 & \mathbf{x}_k^{*T} \end{bmatrix}^T. \end{aligned} \quad (28)$$

Then by (27) the dual of a program equivalent to $\mathcal{D}_{z_{1:k}}(\mathbf{f})$ is

$$\begin{aligned} &\max_{\boldsymbol{\lambda} \in \mathbb{R}^k} \langle \boldsymbol{\lambda}_{k-m+1:k}, \mathbf{f} \rangle \\ &\begin{bmatrix} N_{k \times k} \\ -N_{k \times k} \\ D_{k \times k} \\ -D_{k \times k} \end{bmatrix} \boldsymbol{\lambda} \leq [\mathbf{1}_{4k} + \boldsymbol{\delta} + \boldsymbol{\gamma}]^T. \end{aligned} \quad (29)$$

We now show that this program is equivalent to $\mathcal{P}_{z_{1:k}}(\mathbf{f})$.

Put

$$\mathbf{u} := D_{k \times k} \boldsymbol{\lambda} + \begin{bmatrix} D_U \mathbf{x}_0 \\ 0 \end{bmatrix}; \quad \mathbf{y} := N_{k \times k} \boldsymbol{\lambda} + \begin{bmatrix} N_U \mathbf{x}_0 \\ 0 \end{bmatrix} \quad (30)$$

so

$$\begin{bmatrix} N_{k \times k} \\ -N_{k \times k} \\ D_{k \times k} \\ -D_{k \times k} \end{bmatrix} \boldsymbol{\lambda} - \boldsymbol{\delta}^T = \begin{bmatrix} \mathbf{y} \\ -\mathbf{y} \\ \mathbf{u} \\ -\mathbf{u} \end{bmatrix}. \quad (31)$$

Then there exists $\boldsymbol{\lambda}$ satisfying (30) if and only if \mathbf{u} and \mathbf{y} satisfy

$$-N_{k \times k} \mathbf{u} + D_{k \times k} \mathbf{y} = \begin{bmatrix} B_T \mathbf{x}_0 \\ 0 \end{bmatrix}. \quad (32)$$

To see this, observe that the first m rows of the left hand side of (32) are $-N_L [D_L \boldsymbol{\lambda} + D_U \mathbf{x}_0] + D_L [N_L \boldsymbol{\lambda} + N_U \mathbf{x}_0] = [-N_L D_U + D_L N_U] \mathbf{x}_0 = B_T \mathbf{x}_0$, and the other rows of (32) follow from (1).

Next we show $\langle \boldsymbol{\lambda}_{k-m+1:k}, \mathbf{x}^* \rangle = \langle \mathbf{x}_k(\mathbf{y}, \mathbf{u}), \mathbf{x}^* \rangle$. This is true because

$$\begin{aligned} \mathbf{x}_k(\mathbf{y}, \mathbf{u}) &= (B_T)^{-1} [N_U u_{k-m+1:k} - D_U y_{k-m+1:k}] \\ &= (B_T)^{-1} [N_U D_L \boldsymbol{\lambda}_{k-m+1:k} - D_U N_L \boldsymbol{\lambda}_{k-m+1:k}] \text{ by (30)} \\ &= \boldsymbol{\lambda}_{k-m+1:k} \text{ by (1).} \end{aligned}$$

It remains only to show that the alignment conditions of the Theorem statement hold. The inequalities $|u_j| \leq 1$ and $|y_j - z_j| \leq 1$ follow from (31) and the inequalities (29). The other inequalities in Definition 7 follow directly from the complementary slackness conditions for (27) when the associations (28) are made. ■

Proof of Theorem 9 We are given $\mathbf{x}^- \in S_{k-1}$ and $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$. Since S_{k-1} is non-empty, there exists $(y_{1:k-1}, u_{1:k-1}) \in \arg \max \mathcal{P}_{z_{1:k-1}}(\mathbf{f}^-)$ and $(y_{1:k-1}^*, u_{1:k-1}^*) \in \arg \min \mathcal{D}_{z_{1:k-1}}(\mathbf{f}^-)$, and these sequence pairs are aligned by Theorem 12. They can be extended from length $k-1$ to k using any $(u_k, y_k^*) \in M(\mathbf{x}^-, \mathbf{f}^-, z_k)$, with y_k and u_k^* given by, respectively, (3) and (10). A further application of Theorem 12 shows that $(y_{1:k}, u_{1:k}) \in \arg \max \mathcal{P}_{z_{1:k}}(\mathbf{f})$ and $(y_{1:k}^*, u_{1:k}^*) \in \arg \min \mathcal{D}_{z_{1:k}}(\mathbf{f})$, because $(y_{1:k}, u_{1:k})$ and $(y_{1:k}^*, u_{1:k}^*)$ are feasible for, respectively, $\mathcal{P}_{z_{1:k}}(\mathbf{f})$ and $\mathcal{D}_{z_{1:k}}(\mathbf{f})$, and are aligned. The Theorem statement then follows from (2), (9), Proposition 11, and the necessary and sufficiency condition in Theorem 12. ■

Proof of Theorem 10 By the definition of precursor, $\mathbf{x}^- = \mathbf{x}_{k-1}(y_{1:k}, u_{1:k})$ where $\mathbf{x} = \mathbf{x}_k(y_{1:k}, u_{1:k})$ for some $(y_{1:k}, u_{1:k})$ feasible for $\mathcal{P}_{z_{1:k}}(\cdot)$. From the state space description of the primal system $\mathbf{x} = A\mathbf{x}^- + B\mathbf{u}_k$ and, by Proposition 11, $(y_{1:k}, u_{1:k}) \in \arg \max \mathcal{P}_{z_{1:k}}(\mathbf{f})$ for all $\mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x}_k)$. Now for all $\mathbf{f} \in \mathcal{N}_S(\mathbf{x})$ there exists $(y_{1:k}^*, u_{1:k}^*) \in \arg \min \mathcal{D}_{z_{1:k}}(\mathbf{f})$ and, by the state space description of the dual system, $\mathbf{f} = A^* \mathbf{f}^- + B^* y_k^*$, where $\mathbf{f}^- = \mathbf{x}_{k-1}^*(y_{1:k}, u_{1:k})$. By Theorem 12, $(y_{1:k}, u_{1:k})$ and $(y_{1:k}^*, u_{1:k}^*)$ are aligned. We have shown that $(y_{1:k-1}, u_{1:k-1})$ is feasible for $\mathcal{P}_{z_{1:k-1}}(\mathbf{f}^-)$, and is aligned with $(y_{1:k-1}^*, u_{1:k-1}^*)$, which is feasible for $\mathcal{D}_{z_{1:k-1}}(\mathbf{f}^-)$. By Theorem 12, $(y_{1:k-1}, u_{1:k-1}) \in \arg \max \mathcal{P}_{z_{1:k-1}}(\mathbf{f}^-)$ and finally, from Proposition 11, $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}_{k-1}(y_{1:k-1}, u_{1:k-1}))$, where $\mathbf{x}_{k-1}(y_{1:k-1}, u_{1:k-1}) = \mathbf{x}^-$. ■

Proof of Proposition 14 Let \mathbf{x} be a successor to $\mathbf{x}^- \in \partial S_{k-1}$. The if part is logically equivalent to showing $\mathbf{x} \in \partial S_k \implies M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is nonempty for some $\mathbf{0} \neq \mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$. Now $\mathbf{x} \in \partial S_k$ implies there exists $\mathbf{0} \neq \mathbf{f} \in \mathcal{N}_{S_k}(\mathbf{x})$. By Theorem 10, there exists $\mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ satisfying $\mathbf{f} = A^* \mathbf{f}^- + B^* y_k^*$ and $\mathbf{x} = A \mathbf{x}^- + B u_k$ for some $(u_k, y_k^*) \in M(\mathbf{x}^-, \mathbf{f}^-, z_k)$. We need to show that there exists such \mathbf{f}^- satisfying additionally the condition $\mathbf{f}^- \neq \mathbf{0}$. This is true because if \mathbf{f}^- guaranteed by Theorem 10 satisfies $\mathbf{f}^- = \mathbf{0}$ then it is straightforward to show that there necessarily also exists $\bar{\mathbf{f}}^- \neq \mathbf{0}$ satisfying $\mathbf{f} = A^* \bar{\mathbf{f}}^- + B^* \bar{y}_k^*$ and $\mathbf{x} = A \mathbf{x}^- + B u_k$ for some $(u_k, \bar{y}_k^*) \in M(\mathbf{x}^-, \bar{\mathbf{f}}^-, z_k)$.

The only if part is logically equivalent to the statement $M(\mathbf{x}^-, \mathbf{f}^-, z_k)$ is nonempty for some $\mathbf{0} \neq \mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-) \implies \exists \mathbf{x} \in \partial S_k$, where \mathbf{x} is a successor to \mathbf{x}^- . By Theorem 9, $\mathbf{x} = A \mathbf{x}^- + B u_k \in S_k$ and $\mathbf{f} = A^* \mathbf{f}^- + B^* y_k^* \in \mathcal{N}_{S_k}(\mathbf{x})$ for some $(u_k, y_k^*) \in M(\mathbf{x}^-, \mathbf{f}^-, z_k)$. By varying the magnitude of \mathbf{f}^- it can be shown that the existence of \mathbf{f} satisfying these conditions implies the existence of $\mathbf{f} \neq \mathbf{0}$ satisfying these conditions, which yields $\mathbf{x} \in \partial S_k$ as required. ■

Proof of Proposition 17 All successors \mathbf{x} to \mathbf{x}^- are of the form $A \mathbf{x}^- + B u_k$, where $L(\mathbf{x}^-)$ intersects Q and $(y_k, u_k) \in L(\mathbf{x}^-)$. If \mathbf{x} is not equal to any of $g^T(\mathbf{x}^-)$, $g^B(\mathbf{x}^-)$, $g^L(\mathbf{x}^-)$ and $g^R(\mathbf{x}^-)$ then (y_k, u_k) belongs to the interior of Q , implying there exists $u_k^{(1)}$ and $u_k^{(2)}$, with $u_k^{(1)} < u_k < u_k^{(2)}$, for which $\mathbf{x}^{(1)} := A \mathbf{x}^- + B u_k^{(1)}$ and $\mathbf{x}^{(2)} := A \mathbf{x}^- + B u_k^{(2)}$ are successors to \mathbf{x}^- . Since there exists α , $0 < \alpha < 1$, for which $u_k = \alpha u_k^{(1)} + (1 - \alpha) u_k^{(2)}$, and $\mathbf{x} = \alpha \mathbf{x}^{(1)} + (1 - \alpha) \mathbf{x}^{(2)}$, we have shown that \mathbf{x} is not an extreme point (that is not a vertex) of S_k . ■

Proof of Theorem 24 In Case 1a), $\mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ contains only the zero vector so $\mathbf{x}^- \in \text{int}(S_{k-1})$ and the Theorem statement holds by the results of Section 13.

Case 1b) holds true by Proposition 37, given below.

Case 2a) follows from results in Section 15, and Case 2b) from Section 16.

Proposition 37. Suppose F is a facet of S_k with direction \mathbf{f} , $(\mathbf{f})_m \neq 0$, $\mathbf{x} \in \text{relint}(F)$ and $\mathbf{x}^- \notin \text{int}(S_{k-1})$ is a precursor of \mathbf{x} . Then there is a facet F^- of S_{k-1} whose direction \mathbf{f}^- satisfies $(\mathbf{f}^-)_1 \neq 0$, and $\mathbf{x}^- \in \text{relint}(F^-)$.

Proof. We first show that if F is a facet of S_k with direction \mathbf{f} , $(\mathbf{f})_m \neq 0$, $\mathbf{x} \in \text{relint}(F)$, $\mathbf{x}^- \notin \text{int}(S_{k-1})$ is a precursor of \mathbf{x} , and $\mathbf{0} \neq \mathbf{f}^- \in \mathcal{N}_{S_{k-1}}(\mathbf{x}^-)$ satisfies $(\mathbf{f}^-)_1 = 0$ then $L(\mathbf{x}^-)$ must pass through a corner of Q . This is true because, by (26), $(\mathbf{f})_m = (B^*)_m y_k^*$, so $(\mathbf{f})_m \neq 0 \implies y_k^* \neq 0$, implying also $u_k^* \neq 0$ (because $(\mathbf{f}^-)_1 = 0$). But the only points in Q that are aligned with such (y_k^*, u_k^*) are the corners Q^{NE} and Q^{SW} , implying $\mathbf{x}^- \in H^{\text{NE}}$ or $\mathbf{x}^- \in H^{\text{SW}}$ for any precursors \mathbf{x}^- of any states of F in an open neighbourhood of \mathbf{x} . If for all states \mathbf{x}^- in a facet of S_{k-1} the line $L(\mathbf{x}^-)$ passes through Q^{NE} (or Q^{SW}) then that facet has direction C' , and $(C')_1 \neq 0$ when the slopes of L and L^* have the same sign. This shows $(\mathbf{f}^-)_1 \neq 0$, as required. □

Proof of Theorem 25 It is impossible for a vertex \mathbf{v} of the m -dimensional polytope S_k to be in the intersection of the vertex sets of facets all of whose directions have their last component equal to zero. All facet directions \mathbf{f} in Case 3) of Theorem 24 satisfy $(\mathbf{f})_m = 0$ (that is they are parallel to the x_m axis). Hence any vertex from any facet from Case 3) must be contained in at least one facet from either Case 1) or Case 2) of Theorem 24. We have shown that any vertex of any facet of S_k with direction \mathbf{f} satisfying $(\mathbf{f})_m = 0$ must be a vertex of a facet of S_k with direction \mathbf{f} satisfying $(\mathbf{f})_m \neq 0$. This implies that all the vertices of S_k are vertices of the facets from Case 1) or Case 2) of Theorem 24. By the results of Section 13 the vertices of facets of S_k from Case 1) of Theorem 24 are the vertices from Case 1) of Theorem 25. The vertices of facets from Case 2) of Theorem 24 are precisely the vertices of Case 2) of Theorem 25 by virtue of Propositions 18 and 31. ■

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